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WALSH FUNCTIONS AND GENERALIZED ALMOST CONVERGENCE

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1. Introduction

The Rademacher functions are defined by

$$h_0(x)=1\ \left(0\leq x<\frac{1}{2}\right),\ h_0(x)=-1\ \left(\frac{1}{2}\leq x<1\right),$$

$$h_0(x+1)=h_0(x),\ h_n(x)=h_0(2^n x),\ (n=1, 2, 3, \cdots).$$

The Walsh functions are then given by

$$\phi_0(x)=1,\ \phi_n(x)=h_1(x),\ h_2(x),\ \cdots,\ h_{n_r}(x)$$

for $n=2^{n_1}+2^{n_2}+\cdots+2^{n_r}$, where the integers n_i are uniquely determined by $n_{i+1}<n_i$.

Let $f(x)$ be an integrable function in the sense of Lebesque in [0, 1] and be periodic with period 1. Let the Walsh-Fourier series of $f(x)$ be $\sum_{n=1}^{\infty}a_n\phi_n(x)$, where

$$(1.1)\qquad a_n=\int_0^1 f(x)\phi_n(x)dx.$$

We shall now enumerate important properties and results concerning Walsh-Functions which have been obtained by Fine [1] and which have played a significant role in the theory of Walsh-Fourier series.

For each fixed x and for almost all t, the equation

$$(1.2)\qquad \phi_n(x+1)=\phi_n(x)\phi_n(t)\ \text{holds,}$$

for each fixed x.

$$(1.3)\qquad \int_0^1 f(x+t)dt=\int_0^1 f(t)dt$$

and

$$(1.4)\qquad \int_0^1 f(t)\phi_n(x+t)dt=\int_0^1 f(x+t)\phi_n(t)\ dt.$$

Let

$$J_k(y)=\int_0^y \phi_k(t)dt,\ k=0, 1, 2, \cdots\cdots$$

$$J_k^*(y)=k\ J_k(y).$$

For $k\geq 1$, we write $k=2^n+k^1$, where

$0 \leq k^1 < 2^n$, $n=0, 1, 2, \cdots$, we have also

(1.5) $$J_k(y)=2^{-(n+2)}\{\phi_k 1(y)-\sum_{r=1}^{\infty} 2^{-r}\phi_{2^n+r \atop +k}(y)\}.$$

It is easy to see that

(1.6) $$2^{n+2}J_k(y)=0, \text{ for } y=0,\ 1,$$

and

(1.7) $$|J_k^*(y)| \leq M \text{ for all } y \text{ and } k.$$

Let $B_k(x)$ denote the sequence $\{ka_k\phi_k(x)\}$, where a_k is Walsh-Fourier coefficient of a function of bounded variation.

The matrix $A=(a_{nk})$ is said to be $(c, F_{\mathscr{B}})_{\text{reg}}$ (see [3]) if and only if

(i) $N(A)<\infty$ and there exists a whole number $r \geq 0$ such that
$$\sup \sum_k |\sum_n b_{mn}(i)\ a_{nk}| < \infty,\ 0 \leq i < \infty,\ r \leq m < \infty.$$

(ii) $\lim_m \sum_n b_{mn}(i) a_{nk}=0$ for each k, uniform in i,

(iii) $\lim_m \sum_n b_{mn}(i) \sum_k a_{nk}=1$ uniform in i.

2. We shall prove the following theorem

THEOREM. *If $A \in (c, F_{\mathscr{B}})_{\text{reg}}$, then for every $f \in BV[0, 1]$ and for every $x \in [0, 1]$*
$$\lim_{m\to\infty} \sum_n b_{mn}(i) \sum_k a_{nk}\ B_k(x)=0$$
uniformly in i, if and only if
$$\lim_{m\to\infty} \sum_n b_{mn}(i) \sum_k a_{nk}\ J_k^*(t)=0$$
uniformly in i for every $t \in [\delta, 1]$, $\delta>0$.

PROOF OF THEOREM. We have by virtue of (1. 2) and (1. 4)

$$\sum_k b_{mn}(i)\ \Sigma a_{nk}\ B_k(x)$$
$$=\sum_n b_{mn}(i) \sum_k a_{nk}(ka_k\ \phi_k(x))$$
$$=\sum_n b_{mn}(i) \sum_k a_{nk}\left(k\int_0^1 f(t)\ \phi_k(t)\ \phi_k(x)\ dt\right)$$
$$\sum_n b_{mn}(i) \sum_k a_{nk}\left(k\int_0^1 f(t)\ \phi_k(x+t)\ dt\right)$$
$$=\sum_n b_{mn}(i) \sum_k a_{nk}\left(k\int_0^1 f(t+x)\phi_k(t)\ dt\right)$$

$$=\sum_n b_{mn}(i)\sum_k a_{nk}k[f(x+t)J_k(t)]_0^1-\sum_n b_{mn}(i)\sum_k a_{nk}\int_0^1 df(x+t)J_k^*(t)$$

$$=J_1-J_2, \text{ say.}$$

By virtue of condition (ii) of Theorem S[3], $J_1\to 0$ as $n\to\infty$ uniformly in i. Now, it is sufficient to show that

$$J_2=\int_0^1 df(x+t)K_{n,i}(t)\longrightarrow 0, \text{ as } m\to\infty \tag{2.1}$$

uniformly in i, where

$$K_{n,i}(t)=\sum_n b_{mn}(i)\sum_k a_{nk}J_k^*(t).$$

Now by virtue of condition (i) of Theorem S[3] we have

$$|K_{n,i}(t)|\leq M,\ i=0, 1, 2, \cdots\cdots \tag{2.2}$$

It is easy to show that (2.1) is equivalent to

$$\int_\delta^1 K_{n,i}(t)df(x+t)\longrightarrow 0 \text{ as } m\longrightarrow\infty \tag{2.3}$$

uniformly in i.

Hence by a theorem on the weak convergence of sequences in the Banach space of all continuous function [4, *p.* 134] and the fact that $|K_{n,i}(t)|\leq M$ for all n, i and $t\in[\delta, 1]$, $\delta>0$. Now, this completes the proof of theorem.

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REMARK. If we choose the matrix $\mathscr{B}=\mathscr{B}_0=(I)$, our theorem reduced to Theorem 1 and for $\mathscr{B}=\mathscr{B}_1$ it will become Theorem 2 of A. H. Siddiqi [2].

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