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AN ALTERNATIVE DERIVATION OF FELLER'S NUMBER OF ARRIVALS DISTRIBUTION IN A RENEWAL PROCESS WITH EXPONENTIAL INTERARRIVAL TIMES

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Feller derived the distribution of the number of arrivals in a renewal process with common exponential interarrival times [equation (4.2), p.12, 1]. The distribution of the number of arrivals under the cited stochastic processes forms a Poisson process. Feller gave a number of examples and possible applications.

Feller recognized the fact that his formulation looked like a new derivation of the Poisson distribution [p.12, 1]. What Feller has shown was indeed the relationship between the exponential distribution and the Poisson distribution.

I emphasize this point of view. Examined under this view, Feller's result may be applied to random sampling from a Poisson population utilizing an exponential density or for the purpose of running Monte Carlo experiments in querying problems as in the following simplified manner.

Let T_1, T_2, $\cdots$ be the random variables with the common exponential distribution $\exp\{-x\}$ with the parameter 1. Determine K in which

$$T_1+T_2+\cdots+T_K<\lambda<T_1+T_2+\cdots+T_{K+1}.$$

Then the index K forms a Poisson variate with the parameter λ. Simple! This process a relatively easy method to generate Poisson variates in a computer because the Poisson variates are essentially determined merely by summing the exponential variables.

This simple relationship between the exponential distribution and the Poisson distribution will be more firmly appreciated if we supplement Feller's derivation with a naive approach as in the following.

Let the exponential density represented by $S(t)=\lambda e^{-\lambda t}$ in which the random variates $\{T_1\}$, $i=1, 2, \cdots, k, \cdots$ are assumed to be independently distributed.

We wish to find the distribution of K where K is an index and a random variable defined in $\sum_{i=1}^{k} T_i=C$ and $T_c=0$. Under the assumptions, we have

$$P(K\geq k)=P(\sum_{i=1}^{k} T_i<C)$$

$$=\int\cdots\int\lambda^k \exp\left\{-\lambda\left(\sum_{i=1}^{k} t_i\right)\right\}\prod_{i=1}^{k} dt_i$$
$$\sum_{i=1}^{k} t_i < C$$
$$t_i \geq 0 (i=1, 2, \cdots, k).$$

Set
$$\lambda\sum_{i=1}^{k} t_i = Z_1$$
$$\lambda\sum_{i=2}^{k} t_i = \prod_{i=1}^{2} Z_i$$
$$\vdots$$
$$\lambda t_k = \prod_{i=1}^{k} Z_i.$$

Set $0=\lambda^{-1}$.

We get
$$t_k = \theta\prod_{i=1}^{k} Z_i$$
$$t_{k-1} = \theta\prod_{i=1}^{k-1} Z_i(1-Z_k)$$
$$\vdots$$
$$t_1 = \theta Z_1(1-Z_2).$$

The Jacobian is

$$J=\theta^k \begin{vmatrix} (1-Z_2) & -Z_1 & 0 \quad 0\cdots\cdots0 \\ Z_2(1-Z_3) & Z_1(1-Z_3) & -Z_1Z_2 \; 0\cdots\cdots0 \\ \vdots & \vdots & \vdots \\ \prod_{i=2}^{k-1} Z_i(1-Z_k) & Z_1Z_3\cdots Z_{k-1}(1-Z_k) & -\prod_{i=1}^{k-1} Z_i \\ \prod_{i=2}^{k} Z_i & Z_1Z_3\cdots Z_k & \prod_{i=1}^{k-1} Z_i \end{vmatrix}.$$

The Jacobian is obtained observing the result that the differentiation yields,

$$\frac{\partial t_k}{\partial Z_j} = 0Z_1\cdots Z_{j-1}Z_{j+1}\cdots Z_k$$
$$\frac{\partial t_{k-1}}{\partial Z_j} = 0Z_1\cdots Z_{j-1}Z_{j+1}\cdots Z_{k-1}(1-Z_k)$$
$$(j=1, 2, \cdots, k-1)$$
$$\frac{\partial t_{k-1}}{\partial Z_k} = -0\prod_{i=1}^{k-1} Z_i$$

....

$$\frac{\partial t_1}{\partial Z_1}=0(1-Z_2)$$

$$\frac{\partial t_1}{\partial Z_2}=-0Z_1$$

$$\frac{\partial t_1}{\partial Z_j}=0 \qquad (j=3,\ 4,\ \cdots,\ k).$$

Successive addition of the rows of J, starting from the bottom row yields

$$J=0^k\begin{vmatrix} 1 & 0 & \cdots\cdots & 0 \\ Z_2 & 0 & \cdots\cdots & 0 \\ \vdots & \vdots & & \vdots \\ \prod_{i=2}^{k-1} Z_i & Z_1Z_3\cdots Z_{k-1} & \cdots\cdots & 0 \\ \sum_{i=2}^{k} Z_i & Z_1Z_3\cdots Z_k & \cdots\cdots & 0\prod_{i=1}^{k-1} Z_i \end{vmatrix}$$

$$=\lambda^{-k}\prod_{i=1}^{k-1} Z_i^{k-i}.$$

Therefore

$$P(K\geq k)=\int_0^{\lambda C} e^{-Z_1}Z_1^{k-1}dZ_1\int_0^1 Z_2^{k-2}dZ_2\cdots\int_0^1 Z_{k-1}dZ_{k-1}\int_0^1 dZ_k$$

$$=\frac{1}{(k-1)!}\int_0^{\lambda C} e^{-Z_1}Z_1^{k-1}dZ_1$$

$$=\frac{1}{(k-1)!}\left\{\left[e^{-Z_1}k^{-1}Z_1^k\right]_0^{\lambda C}+k^{-1}\int_0^{\lambda C} Z_1^k e^{-Z_1}dZ_1\right\}$$

$$=\frac{(\lambda C)^k}{k!}e^{-\lambda C}+P(K\geq k+1).$$

Hence

$$P(K=k)=P(K\geq k)-P(K\geq k+1)=\frac{(\lambda C)^k}{k!}e^{-\lambda C}=P(\sum_{i=1}^{k} T_i=C).$$

Therefore K is a Poisson variate with the parameter λC.

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