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INTEGRALS INVOLVING SPHEROIDAL WAVE FUNCTION AND THEIR APPLICATION IN BOUNDARY VALUE PROBLEM OF HEAT CONDUCTION

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1. Introduction

The H-function of several variables has been analogously defined and represented by [12]. For convenience, and brevity, however, we shall use the contracted notation introduced by Srivastava and Panda [12] throughout the present paper.

The known results ([7], p. 16; [4], p. 316, [10], p. 33) required in the sequel may be recalled as follows:

(i) Spheroidal wave function can be expressed as:

$$S_{mn}(c,x)=\sum_{r=0,1}^{\infty}{}^{*} d_r^{mn}(c)P_{m+r}^{m}(x), \tag{1.1}$$

where the coefficients $d_r^{mn}(c)$ satisfy the recursion formula [7, eq. 3.1.4] and the asterisk* over the summation sign indicates that the sum is taken over only even or odd values of r according as $(n-m)$ is even or odd.

(ii) $\int_{-1}^{1}(1-x^2)^{p-1}P_\nu^m(x)dx$

$$=\frac{\pi 2^m\Gamma\left(p+\frac{1}{2}m\right)\Gamma\left(p-\frac{1}{2}m\right)}{\Gamma\left(1+\frac{1}{2}(\nu-m)\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}(\nu+m)\right)\Gamma\left(p-\frac{1}{2}\nu\right)\Gamma\left(1+p+\frac{\nu}{2}\right)}$$

provided that $2Re(p)>|Re(m)|$. (1.2)

(iii) $E_af(a)=f(a+1)$. $E_a^nf(a)=E_a[E_a^{n-1}f(a)]$ (1.3)

where E denotes the finite difference operator. Also, we shall use the following notation throughout the paper:

$$(\alpha)_r=\frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}=\alpha(\alpha+1)\cdots(\alpha+r-1) \tag{1.4}$$

2. Finite integrals

The main integrals to be proved here are the following:

$$J_1(p)=\int_{-1}^{1}(1-x^2)^{p-1}S_{mn}(c, x)H[z_1(1-x^2)^{\sigma_1}, \cdots, z_r(1-x^2)^{\sigma_r}]\,dx$$

$$=2^m\pi\sum_{r=0,1}^{\infty}{}^{*}d_r^{mn}(c)\Big[\Gamma\Big(1+\frac{1}{2}r\Big)\Gamma\Big(\frac{1}{2}-m-\frac{1}{2}r\Big)\Big]^{-1}$$

$$H^{0,\ \lambda+2:\ (\mu', \nu'); \cdots; (\mu^{(r)}, \nu^{(r)})}_{A+2,\ C+2:\ (B', D'); \cdots; (B^{(r)}, D^{(r)})}\left(\begin{matrix}1-p+\frac{1}{2}m : \sigma_1, \cdots, \sigma_r, \\ 1-p+\frac{1}{2}(m+r) : \sigma_1, \cdots, \sigma_r,\end{matrix}\right.$$

$$\Big(1-p-\frac{1}{2}m : \sigma_1, \cdots, \sigma_r\Big) : [(a) : \theta^1, \cdots, \theta^{(r)} : ((b') : \phi'), \ ; \cdots$$

$$\Big(-p-\frac{1}{2}(m+r) : \sigma_1, \cdots, \sigma_r\Big) : [(c) : \Psi', \cdots, \Psi^{(r)}] : ((d') : \delta') :$$

$$\left.\begin{matrix}; [b^{(r)} : \phi^{(r)}] \\ , \cdots, [(d)^{(r)} : \delta^{(r)}]\end{matrix} ; Z_1, \cdots, Z_r\right) \tag{2.1}$$

provided that σ_i are positive numbers such that

$$\Delta_i+\sigma_i>0,\ \ |\arg(Z_i)|<\frac{1}{2}(\Delta_i+\sigma_i)\pi,$$

$$Re(p)>0 \text{ and } Re\Big[p+\sum_{i=1}^{r}\sigma_i\alpha_i\Big]>0,\ \ i=1, \cdots, r,$$

where

$$\Delta_i=\sum_{j=1}^{\lambda}\theta_j^{(i)}-\sum_{j=\lambda+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\nu(i)}\phi_j^{(i)}-\sum_{j=\nu^{(i)}+1}^{B(i)}\phi_j^{(i)}-\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}$$

$$-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0,\ \ i=1, \cdots, r.\ \ \alpha_i=d_j^{(i)}/\delta_j^{(i)},\ \ j=1, \cdots, \mu^{(i)}.$$

The series on the right hand side of (2.1) is convergent.

$$J_2(p)=\int_{-1}^{1}(1-x^2)^{p-1}S_{mn}(c, x)\ {}_uF_v[{}^{\xi_u}_{\eta_v} ; h(1-x^2)^k]\ \cdot$$

$$H[Z_1(1-x^2)^{\sigma_1}; \cdots; Z_r(1-x^2)^{\sigma_r}]\,dx$$

$$=2\pi^m\sum_{r=0,1}^{\infty}{}^{*}\sum_{p'=0}^{\infty}d_r^{mn}(c)\Big[\Gamma\Big(1+\frac{1}{2}r\Big)\Gamma\Big(\frac{1}{2}-m-\frac{1}{2}r\Big)\Big]^{-1}$$

$$\frac{\prod_{j=1}^{u}(\xi_j)\,(h)^{p'}}{\prod_{j=1}^{v}(\eta_j)\ p'!}\ H^{0,\ \lambda+2\ :\ (\mu', \nu'); \cdots; (\mu^{(r)}, \nu^{(r)})}_{A+2,\ C+2\ :\ (B', D'); \cdots; (B^{(r)}, D^{(r)})}$$

$$\begin{pmatrix} [(a):\theta',\cdots,\theta^{(r)}]:\left(1-p-p'k+\frac{1}{2}m:\sigma_i,\cdots,\sigma_r\right), \\ [(c):\Psi',\cdots;\Psi^{(r)}]:\left(1-p-p'k+\frac{1}{2}(m+r):\sigma_i,\cdots;\sigma_r\right), \\ \left(1-p-p'k-\frac{1}{2}m:\sigma_i,\cdots,\sigma_r\right):(b':\phi');\cdots;(b^{(r)},\phi^{(r)}) \\ \left(-p-\frac{1}{2}(m+r)-p'k:\sigma_i,\cdots,\sigma_r\right):(d':\delta');\cdots;(d^{(r)}\ \delta^{(r)}); \\ Z_1,\cdots;Z_r \end{pmatrix} \tag{2.2}$$

where h, k are positive integers (either h or k maybe zero). σ_i are positive numbers such that $\Delta_i+\sigma_i>0$ and

$$|\arg(Z_i)|<\frac{1}{2}(\Delta_i+\sigma_i)\pi,\ Re(p)>0,\ Re(p+\sum_{i=1}^{r}\sigma_i\alpha_i)>0,$$

$i=1,\cdots,r$ and Δ_i, σ_i are given in (2.1). The result (2.2) holds if $u\le v$ ($u=v+1$ and $|h|<1$), none of $\eta_1,\eta_2,\eta_3,\cdots,\eta_v$ is zero or a negative integer with the remaining conditions as stated in (2.1). The series on the right hand side of (2.2) is convergent.

PROOF OF (2.1). To prove (2.1), we first express the spheroidal wave function $S_{mn}(c,x)$ in the series form (1.1), and the terms of multiple contour integrals form [12]. Now, changing the order of integration and summation, evaluating the inner integral with the help of (1.2), and finally reinterpreting the multiple contourintegrals thus involved by the definition of H-function of several variables given by Srivastava and Panda [12], we get the desired result.

Regarding the interchange of the order of integration and summation it is observed that x-integral is convergent if $Re(p)>0$; $Re(p+\sum_{i=1}^{r}\sigma_i\alpha_i)>0$, $i=1,\cdots,r$. The multiple contour integral converges under the conditions stated in (2.1). The series $\sum_{r=0,1}^{\infty}{}^{*}\ d_r^{mn}(c)P_{m+r}^{m}(x)$

converges absolutely and uniformly for all finite x ([7]; 16—17). Hence the interchange of order of integration and summation is justified ([1], p.504).

PROOF OF 2.2. On multiplying both sides of (2.1) by

$\prod_{j=1}^{u}\Gamma(\xi_j+\delta)\ (h)^{\delta}/\prod_{j=1}^{v}\Gamma(\eta_j+\delta)$ and applying the operation $\exp(E^{k}E_{\delta})$ yields.

$$\exp(E^k E_\delta)\Big[J_i(p)\prod_{j=1}^{u}(\xi_j+\delta)h^\delta/\prod_{j=1}^{v}\Gamma(\eta_j+\delta)\Big]$$

$$=2^m\pi\exp(E^k E_\delta)\sum_{r=0,1}^{\infty}{}^{*}d_r^{mn}(c)\Big[\Gamma\Big(1+\frac{1}{2}r\Big)\Gamma\Big(\frac{1}{2}-\frac{1}{2}r-m\Big)\Big]^{-1}$$

$$\frac{\prod_{j=1}^{u}\Gamma(\xi_j+\delta)h^\delta}{\prod_{j=1}^{v}\Gamma(\eta_j+\delta)}H^{0,\lambda+2:(\mu',\nu');\cdots;(\mu^{(r)}\nu^{(r)})}_{A+2,\ C+2:(B',D');\cdots;(B^{(r)},D^{(r)})}\{(T)\}$$

where

$$(\mathrm{T})=\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}]:\Big(1-p+\frac{1}{2}m:\sigma_1;\cdots;\sigma_r\Big),\ \Big(1-p-\frac{1}{2}m:\sigma_1;\cdots;\sigma_r\Big)\\ [(c):\Psi',\cdots,(\Psi)^{(r)}]:\Big(1-p+\frac{1}{2}(m+r):\sigma_1;\cdots;\sigma_r\Big),\ \Big(-p-\frac{1}{2}(m+r)\end{matrix}\right.$$

$$\left.\begin{matrix}:(b',\phi');\cdots;(b^{(r)}:\phi^{(r)})\\ ;\sigma_1,\cdots,\sigma_r):(d',\delta');\cdots;(d^{(r)}:\delta^{(r)}):Z_1,\ \cdots,\ Z_r\end{matrix}\right). \tag{2.3}$$

Taking summation on both sides of (2.3) and using the definition of finite difference operator (1.3), we get

$$\sum_{p'=0}^{\infty}\left\{\frac{\prod_{j=1}^{u}\Gamma(\xi_j+\delta+p')\,(h)^{\delta+p'}}{\prod_{j=1}^{v}(\eta_j+\delta+p')\,p^1!}\int_{-1}^{1}(1-x^2)^{p+p'k-1}\right.$$

$$S_{mn}(c,x)\cdot H[Z_1(1-x^2)^{\sigma_1},\ \cdots,\ Z_r(1-x^2)^{\sigma_r}]\,dx$$

$$=2^m\pi\sum_{p'=0}^{\infty}\sum_{r=0,1}^{\infty}{}^{*}d_r^{mn}(c)\Big[\Gamma\Big(1+\frac{1}{2}r\Big)\Gamma\Big(\frac{1}{2}-\frac{1}{2}r-m\Big)\Big]^{-1}$$

$$\frac{\prod_{j=1}^{u}\Gamma(\xi_j+\delta+p')}{\prod_{j=1}^{v}\Gamma(\eta_j+\delta+p')\,p^1!}H^{0,\lambda+2:(\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})}_{A+2,C+2:(B',D');\cdots;(B^{(r)},D^{(r)})}\{(Q)\}$$

where

$$(\mathrm{Q})=\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}]:\Big(1-p-p'k+\frac{1}{2}m:\sigma_1,\cdots,\sigma_r\Big),\\ [(c):\Psi',\cdots,\Psi^{(r)}]:\Big(1-p-p'k+\frac{1}{2}(m+r):\sigma_1;\cdots,\sigma_r\Big),\end{matrix}\right.$$

$$\left.\begin{matrix}\Big(1-p-p'k-\frac{1}{2}m,\ \sigma_1,\cdots,\sigma_r\Big):(b',\phi'),\cdots,(b^{(r)}:\phi^{(r)})\\ \Big(-p-\frac{1}{2}(m+r)-p'k:\sigma_1,\cdots,\sigma_r):(d',\delta');\cdots,(d^{(r)}:\delta^{(r)}):Z_1;\cdots;Z_r\end{matrix}\right). \tag{2.4}$$

Now changing the order of integration and summation on the left hand side of (2.4) which is justified [2, p. 173], using the result (1.4) and, finally, replacing $\xi_j+\delta$ by ξ_j and $\eta_j+\delta$ by η_j enable us to obtain the value of the integral (2.2).

3. An expansion formula

In this section we derive the following expansion formula:

$$(1-x^2)^{p-1}\,{}_uF_v\left[\begin{matrix}\xi_u\\ \eta_v\end{matrix};h(1-x^2)^k\right]H\,[Z_1(1-x^2)^{\sigma_1};\cdots,Z_r(1-x^2)^{\sigma_r}]$$

$$=\sum_{n=0}^{\infty}J_2(p)\,S_{mn}(c,x),\tag{3.1}$$

which is valid under the same conditions as given in (2.2) with $p\geq 1$. $J_2(p)$ is the value of the integral defined by (2.2). The series on the right hand side of (3.1) is convergent.

PROOF. From the general theory of Sturn-Liouville differential equations, it follows that the function $S_{mn}(c, x)$ form the countably infinite orthonormal set complete in $(-1, 1)$. Hence any arbitrary function $f(x)\in(-1,1)$ can be represented as a linear combination of these functioins, i.e.

$$f(x)=(1-x^2)^{p-1}\,{}_uF_v\left[\begin{matrix}\xi_u\\ \eta_v\end{matrix};h(1-x^2)^k\right]H\,[Z_1(1-x^2)^{\sigma_1};\cdots,Z_r(1-x^2)^{\sigma_r}]$$

$$=\sum_{n=0}^{\infty}A_n\,S_{mn}(c,x),\quad -1<x<1\tag{3.2}$$

(Following Churchil(3) (1963) p. 57, Taylor(13), (1963) p. 111). On multiplying both sides of (3.2) by $S_{mn}(c, x)$, integrating with respect to x over the interval $(-1, 1)$, and making use of the orthogonality property of spheroidal wave functions [7, p. 22 eqs. (3.1, 3.2), (3.1.33)]

$$J_2(p)=A_n\int_{-1}^{1}[S_{mn}(c,x)]^2dx,\ \ for\ n'=n\tag{3.3}$$

because all other terms on the right hand side of (3.3) vanish except for $n'=n$. Now, in order to avoid undesirable consequences in application, we shall normalize the functions $S_{mn}(c, x)$ by the stipulation that

$$\int_{-1}^{1}[S_{mn}(c,x)]^2\,dx=1,\ \ (n-m)\text{ is even or odd}$$

for all values of c.

Hence,

$$A_n=J_2(p). \tag{3.4}$$

Thus, by virtue of (3.2) and (3.4), the desired expansion formula (3.1) follows:

REMARKS. Regarding the convergence of the series on the right hand sides of the results (2.1), (2.2) and (3.1), it would be worth mentioning that the ratio d_{r+2}^{mn}/d_r^{mn} is $-c^2/4r^2$ [6. p. 17] and the ratio of gammas involving r, p' is bounded for large values of r (even or odd) and p' by virtue of the fairly well known result [cf., [5] ; p. 47]

$$\frac{\Gamma(r+\alpha)}{\Gamma(r+\beta)}=r^{\alpha-\beta}[1+0(r^{-1})],\ r\to\infty.$$

Hence the series on the right hand side of (2.1), (2.2) and (3.1) are uniformly and absolutely convergent by M-test.

4. Particular cases

On specializing the parameters of the H-function of several variables of (Srivastava and Panda) in the results (2.1), (2.2) and (3.1), we deduce various known results given earlier by Gupta and Sharma [9], Singh and Verma ([11], p. 325—32).

5. Problem of heat absorption inside the sphere

In this section, the problem of determining of a function $\phi(r, x)$ which represents the temperature inside the non-homogeneous sphere $r\leq a$ is considered. The temperature on the surface $r=a$ is a prescribed function, say $f(x)$, of spherical coordinate 'x' only $(-1\leq x\leq 1)$. Therefore, the fundamental equation of heat conduction is

$$k\left[\frac{\partial^2\phi}{\partial r^2}+\frac{2}{\mathrm{r}}\ \frac{\partial\phi}{\partial r}\right]+\frac{1}{r^2}\ \frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial\phi}{\partial x}\right\}+pc_1\,Q(r,x)=0 \tag{5.1}$$

where $x=\cos\theta$ and $Q(r, x)$ is the sink of heat absorption, and K, p and c, are respectively, the conductivity, density and specific heat of the material of the sphre.

Let $$Q(r,\ x)=-\frac{k}{c_1\,r^2}\left[c^2x^2+\frac{m^2}{1-x^2}\right]\phi$$

which is linearly dependent on the temperature function $\phi(r,x)$. Thus, the equation (5.1) becomes

$$\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\ \frac{\partial\phi}{\partial r}+\frac{1}{r^2}\Big[(1-x^2)\frac{\partial^2}{\partial x^2}-2x\frac{\partial}{\partial x}-c^2x^2-\frac{m^2}{1-x^2}\Big]\phi=0 \qquad (5.2)$$

Under the boundary condition:

$$\phi(a,\ x)=(1-x^2)^{P-1}H\,[Z_1(1-x^2)^{\sigma_1},\ \cdots,\ Z_r(1-x^2)^{\sigma_r}],\ \ -1\leq x\leq 1$$

the solution of (5.2) is given by

$$\phi(r,\ x)=\sum_{n=0}^{\infty}J_1(P)S_{mn}(c,\ x)\Big(\frac{r}{a}\Big)^{\alpha}, \qquad (5\cdot3)$$

where $\alpha=-\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda'_{mn}}$, $J_1(P)$ is the value of the integral defined by (2.1), and the conditions of validity are the same as given in (2.1).

PROOF. To solve the partial differential equation (5.2), we use the 'generalize Legendre transform' recently developed and defined by Gupta [8] as:

$$\bar{f}_{mn}(C)=\int_{-1}^{1}F(x)\ S_{mn}(c,\ x)dx, \qquad (5,4)$$

with the inversion formula

$$F(x)=\sum_{n=0}^{\infty}\frac{\bar{f}_{mn}(c)S_{mn}(c,x)}{N_{mn}}, \qquad (5,5)$$

where N_{mn} is the normalization factor of $S_{mn}(c,\ x)$ given by Flammer [7, 0.22, equ. (3.1.33)]. It is convenient in applications to normalize the function $S_{mn}(c,\ x)$ such that $N_{mn}=1$.

Now, applying the transform (5.4) to equation (5.2), we obtain

$$r^2\frac{\partial^2\bar{\phi}}{\partial r^2}+2r\frac{\partial\bar{\phi}}{\partial r}-\lambda'_{mn}(c)\,\bar{\phi}=0, \qquad (5.6)$$

where

$$\bar{\phi}=\begin{cases}\displaystyle\int_{-1}^{1}\phi(r,\ x)\ S_{mn}(c,\ n)dx & (5.7)\\ J_1(p),\ \text{when } r=a & (5.8)\end{cases}$$

which is bounded in the region $0\leq r\leq a$.

α, β are the roots of the indicial equation obtained after substituting $r=e^z$ in (5.6), thus

$$\left.\begin{aligned}\alpha&=\frac{1}{2}+\frac{1}{2}\sqrt{1+4\lambda'_{mn}(c)}\\ &=-\frac{1}{2}-\frac{1}{2}\sqrt{1+4\lambda'_{mn}(c)},\end{aligned}\right\}. \qquad (5.9)$$

Since the solution corresponding to the root β is inadmissible the solution of equation (5.6) is given by

$$\bar{\phi}=A_1(c)r^{\alpha}. \tag{5.10}$$

In order to determine the coefficient $A_1(c)$, we use the equation (5.8) and get

$$A_1(c)=J_1(p)/a^{\alpha}.$$

Hence substituting the value of $A_1(c)$ in (5.10), we get

$$\bar{\phi}=J_1(p)\left(\frac{r}{a}\right)^{\alpha}.$$

Finally, using the inversion formula (5.5) with the improved convension $N^{mn}=1$, we get the desired solution (5.3).

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