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A NOTE ON SPACES BY EMBEDDINGS IN βX

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1. Introduction

In [3], Raymond F. Gittings defined the following terminologies; a completely regular space X is P_k-embedded in βX, $G_\delta(K)$-embedded in βX, strictly P_k-embedded in βX, strictly $G_\delta(K)$-embedded in βX and strongly P_k-embedded in βX by imposing certain conditions on a space X in terms of the way it is embedded in its Stone-Čech compactification βX.

In this paper we define $w^k M$-spaces as a generalization of $w\Delta$-spaces and wM-spaces and obtain some characterizations of $w^k M$-spaces and strictly P_k-embedded spaces in βX. All completely regular spaces are assumed T_1 and the set of positive integers is denoted by N.

2. Preliminaries and definitions

If A is a subset of a space X, the closure A in X is denoted by $\mathrm{Cl}_x A$. If $\mathscr{U}$ is a collection of subsets of a space X and $x \in X$, we define $\mathrm{St}^k(x, \mathscr{U})$ as follows:

$\mathrm{St}^1(x, \mathscr{U}) = \mathrm{St}(x, \mathscr{U}) = \bigcup \{U \in \mathscr{U} : x \in U\}$,

$\mathrm{St}^k(x, \mathscr{U}) = \bigcup \{U \in \mathscr{U} : U \cap St^{k-1}(x, \mathscr{U}) \neq \phi\}$ for $k \geqq 2$.

If $\mathscr{U}$ and $\mathscr{V}$ are covers of a space X, we write $\mathscr{U} < \mathscr{V}$ if for every $U \in \mathscr{U}$ there exists $V \in \mathscr{V}$ such that $U \subset V$. If $\langle \mathscr{U}_n \rangle$ is a sequence of covers of X such that $\mathscr{U}_{n+1} < \mathscr{U}_n$ for every $n \in N$, then the sequence $\langle \mathscr{U}_n \rangle$ is called a refining sequence.

DEFINITION 2.1 [3]. A completely regular space X is said to be *strictly P_k-embedded* in βX if there exists a refining sequence $\langle \mathscr{B}_n \rangle$ of covers of X by sets open in βX such that

(a) $\bigcap_{n=1}^{\infty} \mathrm{St}^k(x, \mathscr{B}_n) \subset X$ for each $x \in X$ and

(b) for each $x \in X$ and $n \in N$, there exists $n(X) \in N$ such that $\mathrm{Cl}_{\beta x} \mathrm{St}^k(x, \mathscr{B}_{n(x)}) \subset \mathrm{St}^k(x, \mathscr{B}_n)$.

DEFINITION 2.2. A space X is called a *w^kM-space* if there exists a refining sequence $\langle \mathscr{U}_n \rangle$ of open covers of X such that if $x_n \in \mathrm{St}^k(x, \mathscr{U}_n)$, then the sequence $\langle x_n \rangle$ has a cluster point in X.

In this case, the sequence $\langle \mathscr{U}_n \rangle$ will be called w^kM-sequence. In other paper, w^1 M-space is called *$w\Delta$-space* and w^2M-space is called *wM-space.*

EXAMPLE 2.3 (A w^2M-space which is not a w^3M-space). Let $R=[0,\omega]$, $S=[0,\Omega]$ and $T=[0, \Omega]$ with the order topology where ω is the first infinite ordinal and Ω is the first uncountable ordinal. If we put $X=R\times S\times T-\{(\omega,\Omega,\Omega)\}$, then the space X is a w^2M-space but it is not a w^3M-space.

3. Main results

The following lemmas will be used throughout the remainder of paper.

LEMMA 3.1[3]. *Let $\langle \mathscr{B}_n \rangle$ be a sequence of open collections of subsets of βX. If we put $\mathscr{U}_n=\{B\cap X : B\in\mathscr{B}_n\}$ for each $n\in N$, then $\mathrm{St}^k(x,\mathscr{U}_n)=\mathrm{St}^k(x, \mathscr{B}_n)\cap X$ for each $k\in N$.*

LEMMA 3.2[3]. *Let $\langle \mathscr{U}_n \rangle$ be a sequence of open covers of X. If we put $\mathscr{B}_n=\{B$ open in $\beta X : B\cap X\in\mathscr{U}_n\}$ for each $n\in N$, then $\mathrm{St}^k(x, \mathscr{U}_n)=\mathrm{St}^k(x, \mathscr{B}_n)\cap X$ for each $x\in X$.*

LEMMA 3.3[3]. *Let $\langle \mathscr{U}_n \rangle$ be a sequence of open covers of X with $\mathscr{U}_{n+1}<\mathscr{U}_n$ for each $n\in N$. If we put $\mathscr{B}_n=\{B$ open in $\beta X : B\cap X\in\mathscr{U}_n\}$, then $\mathscr{B}_{n+1}<\mathscr{B}_n$ for each n.*

THEOREM 3.4. *A completely regular space X is strictly P_k-embedded in βX for any $k\in N$ if and only if there is a sequence $\langle \mathscr{U}_n \rangle$ of open covers of X satisfying:*

(a) *$P_x=\bigcap_{n=1}^{\infty}\mathrm{St}^k(x,\mathscr{U}_n)$ is a compact subset of X for each $x\in X$;*

(b) *The family $\{\mathrm{St}^k(x,\mathscr{U}_n) : n\in N\}$ is a neighborhood base for the set P_x.*

PROOF. Let $\langle \mathscr{B}_n \rangle$be a refining sequence of covers of X by sets open in βX satisfying (a) and (b) of Definition 2.1. If we put $\mathscr{U}_n=\{B\cap X : B\in\mathscr{B}_n\}$ for each $n\in N$, then by Lemma 3.1, $P_x=\bigcap_{n=1}^{\infty}\mathrm{St}^k(x,\mathscr{B}_n)=\bigcap_{n=1}^{\infty}\mathrm{Cl}_{\beta x}\mathrm{St}^k(x,\mathscr{B}_n)$ is compact subset of X and the family $\{\mathrm{St}^k(x,\mathscr{U}_n) : n\in N\}$ is a neighborhood base for the set P_x.

Conversely, suppose $\langle \mathscr{U}_n \rangle$ is a refining sequence of open covers of X satisfying (a) and (b). If we put $\mathscr{B}_n = \{B$ open in $\beta X : B \cap X \in \mathscr{U}_n\}$ for each $n \in N$, then by Lemma 3.3, $\langle \mathscr{B}_n \rangle$ is a refining sequence of covers of X by sets open in βX and by Lemma 3.2, $\mathrm{Cl}_{\beta x}\mathrm{St}^k(x, \mathscr{B}_{n(x)}) \subset \mathrm{St}^k(x, \mathscr{B}_n)$ for some $n(x) \in N$ and hence $\bigcap_{n=1}^{\infty} \mathrm{St}^k(x, \mathscr{B}_n) \subset X$. Thus X is strictly P_k-embedded in βX.

THEOREM 3.5. *If X is strictly P_k-embedded in βX for any $k \in N$, then X is $w^k M$-space.*

PROOF. If X is strictly P_k-embedded in βX, then there exists a refining sequence $\langle \mathscr{U}_n \rangle$ of open covers of X satisfying (a) and (b) of Theorem 3.4. It is easy to show that $\langle \mathscr{U}_n \rangle$ is a $w^k M$-sequence.

THEOREM 3.6. *A completely regular space X is strictly P_k-embedded in βX for any $k \in N$ if and only if X is a $w^k M$-space having $w^k M$-sequence $\langle \mathscr{U}_n \rangle$ satisfying $\bigcap_{n=1}^{\infty} \mathrm{Cl}_{\beta x}\mathrm{St}^k(x, \mathscr{U}_n) = \bigcap_{n=1}^{\infty} \mathrm{St}^k(x, \mathscr{U}_n)$.*

PROOF. Let $\langle \mathscr{B}_n \rangle$ be a refining sequence of covers of X by sets open in βX satisfying (a) and (b) of Definition 2.1. If we put $\mathscr{U}_n = \{B \cap X : B \in \mathscr{B}_n\}$ for each $n \in N$, then by Theorem 3.5, $\langle \mathscr{U}_n \rangle$ is a $w^k M$-sequence and

$$\bigcap_{n=1}^{\infty} \mathrm{Cl}_{\beta x}\mathrm{St}^k(x, \mathscr{U}_n) = \bigcap_{n=1}^{\infty} \mathrm{St}^k(x, \mathscr{U}_n).$$

Conversely, let $\langle \mathscr{U}_n \rangle$ be a refining sequence of open covers of X such that $\bigcap_{n=1}^{\infty} \mathrm{Cl}_{\beta x}\mathrm{St}^k(x, \mathscr{U}_n) = \bigcap_{n=1}^{\infty} \mathrm{St}^k(x, \mathscr{U}_n)$. Let U be an open set in X containing $\bigcap_{n=1}^{\infty} \mathrm{St}^k(x, \mathscr{U}_n)$ and let W be an open set in βX such that $W \cap X = U$. Consider the set $H_n = \mathrm{Cl}_{\beta x}\mathrm{St}^k(x, \mathscr{U}_n) - W$. It follows that $\mathrm{St}^k(x, \mathscr{U}_n) \subset U$ for some $n \in N$. Thus by Theorem 3.4, X is strictly P_k-embedded.

THEOREM 3.7. *A completely regular space X is strictly P_k-embedded in βX for any $k \in N$ if and only if there exists a refining sequence of covers of X by sets open in βXsuch that $\bigcap_{n=1}^{\infty} \mathrm{Cl}_{\beta x}\mathrm{St}^k(x, \mathscr{B}_n) = \bigcap_{n=1}^{\infty} \mathrm{St}^k(x. \mathscr{B}_n) \subset X$.*

The proof of Theorem 3.7 is analogous to the proof of Theorem 3.6.

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