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A NOTE ON PSEUDOCOMPACT GROUPS

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Introduction

In this note, we characterize pseudocompact groups, by demanding that real valued continuous functions preserve convergent nets in a uniform manner as below.

THEOREM 2.1. *Let (G, τ) be a totally bounded topological group. Then (G, τ) is pseudocompact if and only if whenever a net $\{x_\alpha\}$ converges to x, and f is a real valued continuous function on G and $\varepsilon>0$, there exists* a *d such that $|f(ax_\alpha)-f(ax)|<\varepsilon$ for all $\alpha\geq d$ (independent of a) and for all $a\in G$.*

Incidentally this suggests a characterization of equi-continuity of a family of functions, using nets, which also we furnish.

Section 1. In this section, we collect preliminaries and characterize an equicontinuous family, using nets.

DEFINITION 1.1. Let (G, τ) be a topological group. (G, τ) is said to be *pseudocompact* if every real valued continuous function on G is bounded.

DEFINITION 1.2. Let (G, τ) be a topological group. (G, τ) is said to be *totally bounded,* if for every neighbourhood V of e, we can find a finite set $x_1, \cdots, x_n\in G$, such that $G=\bigcup_{i=1}^{n} x_iV$.

DEFINITION 1.3. Let (G, τ) be a topological group. Let $f:(G, \tau)\to R$ be continuous, f is said to be *uniformly continuous,* if for every $\theta>0$, there exists a neighbourhood V of e such that $|f(x)-f(y)|<\theta$ whenever $x^{-1}y\in V$.

DEFINITION 1.4. Let (X, τ) be a topological space. A family $\mathscr{F}$ of real valued continuous functions on X is said to be *equicontinuous* at $x_0\in X$, if given $\varepsilon>0$, there exists a neighbourhood V of x_0 such that $|f(x)-f(x_0)|<\varepsilon$ whenever $x\in V$ and for all $f\in\mathscr{F}$.

A family of real valued continuous functions on a topological space (X, τ) is said to be *equicontinuous* on X if it is equicontinuous at all $x\in X$.

THEOREM 1.5. *Let (X, τ) be a topological space. Let $\mathscr{F}$ be a family of real valued continuous functions on X. Then the following are equivalent*:

(i) *$\mathscr{F}$ is equicontinuous on (X, τ).*

(ii) *Whenever a net $\{x_\alpha\}$ converges to x, and $\varepsilon>0$, there exists a d such that $|f(x_\alpha)-f(x)|<\varepsilon$ for all $\alpha\geq d$ (independent of f) and for all $f\in\mathscr{F}$.*

PROOF. (i)$\Longrightarrow$(ii): Since $\mathscr{F}$ is equicontinuous at x, by 1.4, there exists a neighbourhood V of x such that $|f(x)-f(y)|<\varepsilon$ for all $y\in V$ and for all $f\in\mathscr{F}$. Since $\{x_\alpha\}$ converges to x, there is a $d\in D$ such that $x_\alpha\in V$ for all $\alpha\geq d$. Hence (ii) follows easily.

To show (ii)$\Longrightarrow$(i), suppose (i) does not hold. Then there is at least one $x_0\in X$, at which $\mathscr{F}$ is not equicontinuous. That is we can find an $\varepsilon>0$ such that no neighbourhood of x_0 will satisfy the condition of equicontinuity. So let V be a neighbourhood of x_0. Then there exists a function $f_V\in\mathscr{F}$ and an element x_V of V such that $|f_V(x_V)-f_V(x_0)|\geq\varepsilon$. Now consider the directed set D of neighbourhoods of x_0 directed by $U\geq V$ if $U\subset V$. We have a net $V\rightarrow x_V$ for $V\in D$. This net easily converges to x_0. For this net, ε and $\mathscr{F}$, the hypothesis of (ii) is violated. This contradiction yields the result.

Section 2. In this section we prove the main result.

THEOREM 2.1. *Let (G, τ) be a totally bounded topological group. Then (G, τ) is pseudocompact if and only if whenever a net $\{x_\alpha\}$ converges to x, and f is a real valued continuous function on G and $\varepsilon>0$, there exists a d, such that $|f(ax_\alpha)-f(ax)|<\varepsilon$ for all $\alpha\geq d$(independent of a) and for all $a\in G$.*

PROOF. Let (G, τ) be pseudocompact. Let a net $\{x_\alpha\}$ converge to x, and f be a real valued continuous function on G and $\varepsilon>0$. Then f is (left) uniformly continuous by Theorem 1.5 of [1]. Hence there exists a symmetric neighbourhood V of e such that $|f(x)-f(y)|<\varepsilon$ whenever $x^{-1}y\in V$. Since $\{x_\alpha\}$ converges to x, $\{x^{-1}x_\alpha\}$ converges to e. Hence there is a d such that $x^{-1}x_\alpha\in V$ for all $\alpha\geqslant d$. Now $(ax)^{-1}(ax_\alpha)=x^{-1}x_\alpha\in V$ for all $\alpha\geq d$ (independent of a). Thus it follows that $|f(ax)-f(ax_\alpha)|<\varepsilon$ for all $\alpha\geqslant d$ independent of a and for all $a\in G$.

Let now (G, τ) satisfy the condition in Theorem 2.1. Let f be any real valued continuous function on G. We assert that f is (left) uniformly continuous. Let $\varepsilon>0$ be given. Consider now the family $\mathscr{F}=\{f_a|a\in G\}$ where f_a is a function defined from G to R by $f_a(x)=f(ax)$. The condition in Theorem 2.1 applied to

f, any net $\{x_\alpha\}$ converging to x and $\varepsilon > 0$ implies that $\mathscr{F}$ satisfies the condition (ii) in Theorem 1.4. Hence by Theorem 1.4 there exists a neighbourhood V of e such that $|f_a(x) - f_a(e)| < \varepsilon$ for all $x \in V$ and all $a \in G$. Let now $x^{-1}y \in V$. Then $|f_x(x^{-1}) - f_x(e)| < \varepsilon$. This shows that $|f(y) - f(x)| < \varepsilon$ whenever $x^{-1}y \in V$. Hence f is (left) uniformly continuous. But Theorem 4.1 ([1]) asserts that if G is totally bounded and each real valued continuous function from G to R is left uniformly continuous then G is pseudocompact. Hence the theorem follows.

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REFERENCE

[1] Comfort and Ross, *Pseudocompactness and uniform continuity in topological groups,* Pacific Jour. Math. Vol. 16, No. 3, 1866.