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SOME BRIEF COMMENTS CONCERNING GENERALIZED ZEROS IN SEMIGROUPS

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1. Introduction

An element z of a semigroup S is called a *generalized zero, g-zero,* if and only if for all a, $b \in S$ it follows that $azb=bza$. Clearly, any zero of S is a g-zero. In a commutative semigroup, every element is a g-zero. If a semigroup has an identity that is also a g-zero, then the semigroup is commutative. In particular, a group is commutative if and only if every element in the group is a g-zero.

2. Some basic properties

With the exception of the above definition the definitions used will be standard and can be found in [1], [2] or [3]. The following result justifies the term generalized zero.

PROPOSITION 1. *If a semigroup S has a unique g-zero z, then z is a zero of S.*

PROOF. Let $a \in S$ and consider az. For any other $c, d \in S$ it follows that

$$c(az)d=(ca)zd=dz(ca)=(dzc)a=(czd)a=cz(da)=(da)zc=d(az)c.$$

Hence, az is a g-zero of S. Likewise, za is a g-zero of S. Since S has a unique g-zero, it follows that z is a zero of S.

COROLLARY 1. For any semigroup S, let $K(S)$ denote the collection of all g-zeros of S. Then $K(S)$ is empty or is a two-sided ideal of S.

As expected from the definition and as mentioned earlier, the existence of a g-zero in a semigroup very often has some important applications concerning commutativity.

PROPOSITION 2. *Let S be a semigroup and let T be the subset of elements of $K(S)$ that divide every element of S on the right. Then the elements of T divide the elements of S on the left. In fact either $T=\phi$ or T is a group.*

PROOF. Let $z \in T$ and $a \in S$. Then there exists $b, c, d \in S$ such that $a=bz$,

$b=cz$ and $c=dz$. Thus,

$$a=bz=czz=dzzz=zdzz=zcz=zb.$$

Moreover, when $T\neq\phi$, any element of $K(S)$ that divides all the other elements of $K(S)$ on the right must also divide all the elements of S on the right. The proof follows from Corollary 1 since the set of elements of $K(S)$ that are both right and left divisors of all elements of $K(S)$ is either empty or a group.

COROLLARY 2. *If a semigroup S has a g-zero, z, that divides every element on the right, then the semigroup is commutative.*

Clearly any idempotent g-zero of S commutes with every element of S. Also, if any g-zero of S is regular, then S contains an idempotent g-zero. The next result concerns idempotent g-zeros in a rectangular semigroup [5].

PROPOSITION 3. *Let S be a rectangular semigroup with an idempotent g-zero. Then S is commutative.*

PROOF. Let z be the idempotent g-zero. Since S is rectangular and z is idempotent it is known that $azb=ab$ for every $a, b\in S$. The result is now immediate.

It should be noted that any nontrivial rectangular semigroup S containing g-zeros must contain a g-zero that is not idempotent. For otherwise it is known that if x and y are idempotent and $xy=yx$, then $x=y$. Thus, if all the g-zeros were idempotent then by Proposition 1, S would contain a zero z. But then $ab=azb=0$ for all $a, b\in S$.

We conclude this paragraph with some comments concerning regular g-zeros.

PROPOSITION 4. *Let z be a regular g-zero in a semigroup S. Then z has a unique regular conjugate.*

PROOF. Assume that x and y are both regular conjugates of z. Then

$$xzy=xzxzy=xzyzx=xzx=x.$$

Likewise, $yzx=y$. Hence, $x=xzy=yzx=y$.

Thus, when all the g-zeros of a semigroup are regular, then $K(S)$ has the important property that $K(S)$ is an inverse subsemigroup of S. In particular,

COROLLARY 3. *In a regular semigroup S with g-zeros, $K(S)$ is an inverse semigroup.*

McLean [4] has shown that all the elements of a semigroup S are mutually regularly conjugate if and only if $a, b \in S$ and $a \neq b$ always implies that $ab \neq ba$. Thus, any semigroup in which all the elements are mutually regularly conjugate and which contains a g-zero z can contain only one element. This follows since for all $a, b \in S$ it follows easily that $ab = azb = bza = ba$. This leads to the following proposition.

PROPOSITION 5. *Let S be a semigroup containing g-zeros. If all the elements of $K(S)$ are mutually regularly conjugate, then S contains a zero.*

PROOF. The proof follows from the above remarks and Proposition 1.

3. Simple and 0-simple semigroups with g-zeros

Since a semigroup S without a zero is simple when it contains no proper two-sided ideals, it follows from Corollary 1 that a simple semigroup S with g-zeros is a semigroup without a zero for which $K(S) = S$. Also, a semigroup S with a zero, 0, is 0-simple provided that $S^2 \neq \{0\}$ and the only two-sided ideals are $\{0\}$ and S. Thus, for a 0-simple semigroup S with g-zeros it follows that either $K(S) = \{0\}$ or $K(S) = S$. When $K(S) = S$, S has a special structure.

PROPOSITION 6. *Any 0-simple semigroup S that contains a nonzero g-zero is commutative.*

PROOF. Since S is 0-simple it follows that $S = SzS$ for every $z \in S$ [2, p. 58]. In particular let $z \in K(S) \setminus \{0\}$. Let $x, y \in S$. Then there exists $a, b, c, d \in S$ such that $x = azb$ and $y = czd$. Thus, using Corollary 1 several times gives

$$\begin{aligned} xy &= (azb)(czd) = a(zbcz)d = d(zbcz)a = (dzb)(cza) = (bzd)cza \\ &= (b(zd)c)za = (c(zd)b)za = (czd)(bza) = yx, \end{aligned}$$

i.e., S is commutative.

For simple semigroups with g-zeros we have the following.

PROPOSITION 7. *Any simple semigroup S that contains a g-zero is a commutative group.*

PROOF. As in Proposition 6, S is commutative. Hence, any left or right ideal is a two-sided ideal. Since S is simple it follows that S does not contain any proper left or right ideals. Therefore, it follows immediately that S is a group.

Finally a similar result holds for a semigroup S without a zero if we weaken the requirement that S be simple to the requirement that S not contain any proper left (or right) ideals.

PROPOSITION 8. *Any semigroup S that does not contain any proper left ideals and that does contain a g-zero z is a commutative group.*

PROOF. Since Sz is a left ideal it follows that $S=Sz$, i.e., z divides every element of S on the right. From Corollary 2 it follows that S is commutative. Thus any right ideal is also a left ideal. Since S does not contain any proper left ideals, it does not contain any proper right ideals. It follows that S is a group.

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