Kyungpook Math. J.
Volume 21, Number 2
December, 1981

REGULAR GENERAL CONTACT MANIFOLDS

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1. Introduction

It has proved that a compact connected manifold M^{2n+s} with a regular normal f-structure is the bundle space a principal T^s-bundle over a complex manifold N^{2n}. Moreover, if M^{2n+s} is a K-manifold, then N^{2n} is a Kaehler manifold, [2]. In this work we prove that (Theorem 4.1) if the K-structure on M^{2n+s} is an S-structure, then N^{2n} is a Hodge manifold. Conversely (Theorem 4.4), given a Hodge manifold $N^{2n}$and any $s\geqslant 1$, there exists a principal toroidal bundle $M(N, T^s)$ over N, whose bundle space M^{2n+s} has a regular S-structure.

2. Normal f-structures

A C^{∞}-manifold M^{2n+s}, $n\geqslant 1$, is said to have an *f-structure*, if the structural group of its tangent bundle is reducible to $U(n)\times O(s)$. This is equivalent to the existence of a tensor field on M of type (1,1), rank $2n$, satisfying $f^3+f=0$. Almost complex structures ($s=0$) and almost contact structures ($s=1$) are two examples of f-structures. If there exist vector fields E_i and 1-forms, η^i, $1\leqslant i\leqslant s$ such that

$$f(E_i)=0,\ \ \eta^i(E_j)=\delta^i_j,\ \ \eta^i\circ f=0,\ \ f^2=-I+\sum_{i=1}^{s}\eta^i\otimes E_i$$

we say that M^{2n+s} has a framed f-structure, or, simply an (f, E_i, η^i)-structure. A framed f-structure is normal if

$$S=[f, f]+\sum_{i=1}^{s}d\eta^i\otimes E_i$$

vanishes, where $[f, f]$ is the Nijenhuis tensor of g. In this case we have [3]:

1) $L_{E_i}\eta^j=0$, 2) $[E_i, E_j]=0$, 3) $L_{E_i}f=0$, 4) $d\eta^i(fX, Y)=-d\eta^i(X, fY)$.

The equality 2) implies that the vertical distribution (the one generated by all the E_i) is integrable.

It is known that for any (f, E_i, η^i)-structure there exists a Riemannian metric g which satisfies

$$g(X, Y)=g(fX, fY)+\sum_{i=1}^{s}\eta^i(X)\eta^i(Y).$$

A framed f-str ucture together with this metric is called a *framed metric f-structure*, or, *simple*, an, *(f, E_i, η^i, g)-structure*. The 2-form

$$F(X, Y)=g(X, fY)$$

is called the *fundamental 2-form of the (f, E_i, η^i, g)-structure*. A K-structure is a normal (f, E_i, η^i, g)-structure whose fundamental 2-form is closed.

Let D be an integrable distribution of dimension h on a manifold N^m. A cubical coordinate neighborhood $(U, (x^1, \cdots, x^m))$ on N^m is said to be *regular* with respect to D if $\frac{\partial}{\partial x'}, \cdots, \frac{\partial}{\partial x^h}$ is a basis for $D(p)$, for every $p\in U$, and if each leat of D intersects U in at most one n-dimensional slice of $(U, (x', \cdots, x^m))$. We call D regular if each point $p\in N$ has a cubical coordinate neighborhood which is regular with respect to D.

An(f, E_i, η^i)-structure is said to be *regular* if the vertical distribution is integrable and regular, and if each E_i is regular (the distribution generated by E_i is regular).

Let's state the theorem mentioned at the begining:

THEOREM 2.1 (Blair, Ludden, Yano). *Let M^{2n+s}, $n\geq 1$, be a compact connected manifold with a regular framed f-structure. Then M^{2n+s} is the bundle space of a principal toroidal bundle over a complex manifold N^{2n}. Moreover, if the framed f-structure is a K-structure, then N^{2n} is a Kaehler manifold.*

3. Toroidal bundles

Let $T^1=S^1$ and $T^s=\underbrace{S^1\times\cdots\times S^1}_{s}$ be the one-dimensional and s-dimensional torus respectively. Since these Lie groups are commutative, by choosing A, a nonzero element of the Lie algebra $L(T^1)$ of T^1, we identify $L(T^1)$ with R, and $L(T^s)=L(T^1)\times\cdots\times L(T^1)$ with R^s by means of

$$(0, \cdots, \underset{i}{A}, 0, \cdots, 0)\longleftrightarrow e_i,$$

where $e_1, \cdots, e_s$ is the canonical basis of R^s.

Let $P[N, T^s]$ be the set of all T^s-bundles over the manifold N. If $P(N, T^s, \pi)$ and $Q(N, T^s, \pi)$ are two elements in this set, on

$$\Delta(P\times Q)=\{(u,v)\in P\times Q\,|\,\pi(u)=\pi'(v)\}$$

we define the equivalent relation:

$(u_1, v_1)\sim(u_2, v_2)\Leftrightarrow t\in T^s$ such that $(u_1t, v_1t^{-1})=(u_2, v_2)$. The action of T^s on $\Delta(P\times Q)$ given by $((u, v), t)\to(ut, v)$, induces an action of T^s on

$$P+Q=\frac{\Delta(P\times Q)}{\sim}$$

obtaining, in this way, the new T^s-bundle $P+Q$. It is known that $P[N, T^s]$ with this operation, "+", is an abelian group whose identity element is the trivial bundle $N\times T^s$, [4].

If ω is a connection form with curvature form Ω of a bundle $P(N, T^s)$, then

$$\omega=\sum_{i=1}^{s}\omega_i\otimes e_i \text{ and } \Omega=\sum_{i=1}^{s}d\omega_i\otimes e_i.$$

Each real 2-form $d\omega_i$ is horizontal and right invariant, therefore there exists a unique real 2-form Ω_i^* on N satisfying $d\omega_i=\pi^*\,\Omega_i^*$. Since the forms Ω_i^* are closed, they determine s cohomology classes $[\Omega_i^*]$, $1\leqslant i\leqslant s$ in $H^2(N,R)$. These cohomology classes are independent from the connection. In this way we get the function

$$\Psi: P[N, T^s]\to\bigoplus_{i=1}^{s}H^2(N, R) \text{ given by } P\to([\Omega_1^*], \cdots, [\Omega_s^*]).$$

Our intention now is to show that Ψ is a group homomorphism.

Suppose that $\{\phi_{\beta\alpha}\}$ are the transition function of $P(N, T^s)$ corresponding to some covering $\{U_\alpha\}$. Each function $\phi_{\beta\alpha}: U_\beta\cap U_\alpha\to T^s$ can be written as

$$(\phi^1_{\beta\alpha}, \cdots, \phi^s_{\beta\alpha}).$$

Now $\{\phi^i_{\beta\alpha}\}$ are the transition functions of a 1-dimensional toroidal bundle P_i over N. If we construct the whitney sum $P_1\oplus\cdots\oplus P_s$, it happens that a set of transition functions of this sum is precisely $\{\phi_{\beta\alpha}\}$. In other words, P and $P_1\oplus\cdots\oplus P_s$ have the same transition function. Therefore we may assume that

$$P=P_1\oplus\cdots\oplus P_s \text{ and } P[N, T^s]=\bigoplus_{i=1}^{s}P[N, T^1].$$

Let h_i be the projection $h_i: P_1\oplus\cdots\oplus P_s\to P_i$. If Ω_i is a curvature form on P_i, there is a connection on P whose curvature form Ω satisfies:

$$\Omega=\sum_{i=1}^{s}h_i^*\,\Omega_i\otimes e_i.$$

Therefore we can assume that the function

$$\Psi : P[N, T^s] = \overset{s}{\underset{i=1}{\oplus}} P[N, T^1] \rightarrow \overset{s}{\underset{i=1}{\oplus}} H^2(N, R)$$

is given by $\underbrace{\Psi = \Psi \times \cdots \times \Psi}_{s}$ where Ψ is the function

$$\Psi : P[N, T^1] \rightarrow H^2(N, R) \text{ such that } \Psi(P_i) = [\Omega_i^*].$$

But this Ψ is precisely the function defined by S. Kobayashi in page 32 of [4]. Furthermore, he proves that $\Psi : P[N, T^1] \rightarrow H^2(N, R)$ is a group homomorphism which sends $P(N, T^1)$ onto $H^2(N, Z)_b$, where $H^2(N, Z)_b$ is the subgroup of $H^2(N, R)$ formed by all the elements which contain an integral closed from. Therefore

THEOREM 3.1. *The function*

$$\Psi : P[N, T^s] \rightarrow \overset{s}{\underset{i=1}{\oplus}} H^2(N, R)$$

$$P \rightarrow ([\Omega_1^*], \cdots, [\Omega_s^*])$$

is a group homomorphism, which sends $P[N, T^s]$ *onto*

$$\overset{s}{\underset{i=1}{\oplus}} H^2(N, Z)_b.$$

4. Regular S-structures

DEFINITION. A manifold M^{2n+s} is said to have an *s-contact structure* if there exist on M s global, linearly independent 1-forms $\eta^1, \cdots, \eta^s$ such that $d\eta^1 = \cdots = d\eta^i$, $d\eta^i$ has rank 2^n and, at every point of M,

$$\eta^1 \wedge \cdots \wedge \eta^s \wedge (d\eta^i)^n \neq 0.$$

It is known [1] that if M^{2n+s} has s-contact structure, then it has an (f, E_i, η^i, g)-structure, which we call associated to the s-contact structure, such that $F = d\eta^i$, where F is the fundamental 2-form. A normal (f, E_i, η^i, g)-structure associated to an s-contact structure is called an *S-structure*. Notice that an S-structure is a K-structure.

THEOREM 4.1. *Let* M^{2n+s} *be a compact connected manifold with a regular* S*-structure* (f, E_i, η^i, g), $i=1, \cdots, s$. *Then* M^{2n+s} *is the bundle space of a principal toroidal bundle over a Hodge manifold* N^{2n}.

PROOF. By Theorem 2.1 and its proof we have that $M^{2n\pm s}$ is the bundle space of a principal T^s-bundle over a Kaehler manifold N^{2n}, and that the group action is given by the one-parameter groups of transformations of the vector fields $E_1, \cdots, E_s$.

Now we claim that the form

$$\omega=\sum_{i=1}^{s}\eta^i\otimes e_i$$

is a connection form. This is, ω satisfies:

a) $R_t^*\omega=\omega$, for $t\in T^s$.

b) $\omega(X^*)=X$, where X^* is the fundamental vector fields of X, with X in the Lie algebra of T^s.

Part a) follows from the fact $L_{E_i}\eta^j=0$, $i, j=1, \cdots, s_j$, which is a consequence of the normality of the S-structure. For part b) it suffices to prove it for the vector e_i, $i=1, \cdots, s$. But this follows immediately from $e_i^*=E_i$.

On the other hand, from the proof of Theorem 2.1, we also have that the fundamental from of the f-structure, F, and the fundamental for of the Kaehlerian structure, Ω^*, are related by

$$F=\pi^*\Omega^*$$

where π is bundle projection. But, in the particular case of an S-structure, we have $F=d\eta^i$, $i=1, \cdots, s$. Therefore $d\eta^i=\pi^*\Omega^*$. Hence, by Theorem 3.1, $[\Omega^*]$ is $H(N, Z)_b$, which says that N^{2n} is a Hogde manifold.

THEOREM 4.2. *Let $M(N, T^s, \pi)$ be a principal toroidal bundle whose base space N^{2n} has an almost Hermitian structure. Then M has a regular (f, E_i, η^i, g)-structure, $i=1, \cdots, s$.*

PROOF. Fix a connection form $\omega=\sum_{i=1}^{s}\eta^i\otimes e_i$ on M and let E_i be the fundamental vector of e_i, $1\leqslant i\leqslant s$. Then we have

$$\eta^i(E_j)=\delta^i_j.$$

Let (J, g') be the almost Hermitian structure of N. If $u\in M$, $\pi(u)=v$ and $\bar{\pi}_v: T_v(N)\to T_u(M)$ is the lifting with respect to the fixed connection, define f by

$$f(X)=(\bar{\pi}_v\circ j\circ\pi_u)(X),\ \ X\in T_u(M).$$

Then we have $f(E_i)=0$ and $\eta^i \circ f=0$, $i=1, \cdots, s$. We also have

$$f^2(X)=(\bar{\pi} \circ j \circ \pi)^2(X)=-(\bar{\pi} \circ \pi)(X)=-X+\sum_{i=1}^{s}\eta^i(X)E_i$$

this is, $f^2=-I+\sum_{i=1}^{s}\eta^i \otimes E_i$. Thus we have an (f, E_i, η^i)-structure, $1 \leqq i \leqq s$, on M. Furthermore, the Riemannian metric g on M defined by

$$g(X, Y)=g'(\pi X, \pi Y)+\sum_{i=1}^{s}\eta^i(X)\eta^i(Y)$$

is associated to this (f, E_i, η^i)-structure, since

$$\begin{aligned} g(fX, fY) &= g'(\pi fX, \pi fY)+\sum_{i=1}^{s}\eta^i(fX)\eta^i(fY) \\ &= g'(J\pi X, J\pi Y)=g'(\pi X, \pi Y) \\ &= g(X, Y)-\sum_{i=1}^{s}\eta^i(X)\eta^i(Y). \end{aligned}$$

It is clear from the definition of E_i that each one of these is regular. The regularity of the distribution determined by all the E_i's (vertical distribution) follows from the Theorem XIV of [5], which says that if the leaf space of an integral distribution is a manifold and if the projection mapping takes the tangent space of any point onto the tangent space of its projection, then the distribution must be regular.

THEOREM 4.3. *The framed f-structure defined in the previous theorem is normal if and only if the following two conditions hold:*

1) *J is a complex structure.*
2) *$d\omega(fX, Y)=-d\omega(X, fY)$, for any X, Y.*

PROOF. Since 2) is equivalent to 3) $d\omega(fX, fY)=d\omega(X, Y)$ the theorem will follow as soon as we prove the two equalities:

a) $\pi(S(X, Y))=[J, J](\pi X, \pi Y)$; X, Y right invariant vector fields.

b) $\omega(SX, Y))=d\omega(X, Y)-d\omega(fX, fY)$, for any X, Y.

a) If X, Y are right invariant vector fields on M, so are $[X, Y]$, $f(X)$ and $f(Y)$. (f is right invariant). Besides, we have the relations:

$$\pi[X, Y]=[\pi X, \pi Y] \text{ and } \pi \circ f=J \circ \pi.$$

Therefore

$$\pi(S(X, Y))=\pi([f, f](X, Y)+\Sigma d\eta^i(X, Y)E_i)=[J, J](\pi X, \pi Y).$$

b) Since f is horizontal we have $d\omega(fX, fY)=-\omega([fX, fY])$. Hence

$$\omega(S(X, Y))=\omega([fX, fY])+d\omega(fX, fY)+d\omega(X, Y).$$

THEOREM 4.4. *Let N^{2n} be a Hodge manifold. Then for each $s\geqslant 1$ there exists a principal toroidal bundle $M(N, T^s, \pi)$, whose bundle space M^{2n+s} has a regular S-structure.*

PROOF. Let (J, g') be the Hodge structure on N, and Ω^* its fundamental 2-form. Since $[\Omega^*]\in H^2(N, Z)_b$, then

$$(\underbrace{[\Omega^*], \cdots, [\Omega^*]}_{s})\in \bigoplus_{i=1}^{s} H^2(N, Z)_b.$$

By Theorem 3. 1, there exists a toroidal bundle $M=M(N, T^s, \pi)$ such that $\Psi(M)=([\Omega^*], \cdots, [\Omega^*])$. We can find a connection form $\omega=\sum_{i=1}^{s}\eta^i\otimes e_i$ whose curvature from $d\omega$ satisfies

$$d\omega=\sum_{i=1}^{s}d\eta^i\otimes e_i=\sum_{i=1}^{s}\pi^*\Omega^*\otimes e_i.$$

The forms $\eta^1, \cdots, \eta^s$ define a s-contact structure on M^{2n+s}. In fact, since $d\eta^i=\pi^*\Omega^*$, the ranh of $d\eta^i$ is $2n$.

On the other hand, if $E_1, \cdots, E_s$ are the fundamental vector fields of $e_1, \cdots, e_s$, we have $\eta^i(E_j)=\delta^i_j$. Now, taking $E_1, \cdots, E_s$ and $X_1, \cdots, X_{2n}$ horizontal and linearly independent vectors, we get

$$\begin{aligned}&\eta^1\wedge\cdots\wedge\eta^s\wedge(d\eta^i)^n(E_1, \cdots, E_s, X_1, \cdots, X_{2n})\\&=(d\eta^i)^n(X_1, \cdots, X_{2n})=\Omega^*(\pi X, \cdots, \pi X_{2n})\neq 0\end{aligned}$$

which proves that $\eta^1\wedge\cdots\wedge\eta^s\wedge(d\eta^i)^n\neq 0$ at every point of M.

If (f, E_i, η^i, g) is the framed f-structure on M constructed in the Theorem 4.2 using the Hodge structure (J, g′) on N, we have

$$F(X, Y)=g(X, fY)=g'(\pi X, \pi fY)=g'(\pi X, J\pi Y)=\Omega^*(\pi X, \pi Y)=d\eta^i(X, Y).$$

Therefore this $(f, E_i\,\eta^i, g)$-structure is associated to this s-contact structure defined by $\eta^1, \cdots, \eta^s$. By Theorem 4.2 and its proof, $(f, E_i\,\eta^i, g)$ is regular. On the other hand, since J is a complex structure and $d\omega(fX, fY)=d\omega(X, Y)$, (f, E_i, η^i, g) is normal, and therefore a regular S-structure on M.

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REFERENCES

[1] Blair, D., *Geometric of manifolds with structural group $U(n)\times O(s)$*, Journal of Differential Geometric, 4(1970), pp. 155—167.

[2] ________ , Jano, K. Ludden, G., *Differential geometric Structures on principal roroidal bundles*, Transaction of the American Mathematical Society, 181(1973), pp. 175—184.

[3] Goldberg, S. I. and Yano, K., *On normal globally framed manifolds*, Tohoku Mathematical Journal, 22(1970), pp. 362—370.

[4] Kobayashi, S., *Principal toroidal bundles with 1-dimensional toroidal group*, Tohoku Mathematical Journal, 8(1956), pp. 29—45.

[5] Palais, R., *Grobal formulation of the Lie theory of transformation groups*, Memoirs of the American Mathematical Society, No 22, (1957).