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ON THE $|\overline{N}, p_n|$-SUMMABILITY FACTORS OF INFINITE SERIES

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The result of this paper concerns with the $|\overline{N}, p_n|$-summability of the factored series $\Sigma a_n\varepsilon_n/\theta_n$ whenever the $(C,1)$-mean of the sequence $|\mu_n|$ [μ_n, being the $(\overline{N}, p_n)$-mean of the sequence $\{a_nP_n/p_n\}$] is of certain order γ_n, with certain conditions on ε_n, θ_n, γ_n, and the generating sequence of coefficients $\{p_n\}$. The result generalizes a number of previously known results in this line.

1. Let Σa_n be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real constants such that P_n tends to infinity with n, where

$$P_n=p_0+p_1+p_2+\cdots+p_n\,;\, P_{-1}=p_{-1}=0.$$

The sequence-to-sequence transformation:

$$\bar{t}_n=(P_n)^{-1}\sum_{\nu=0}^{n} p_\nu\, s_\nu, \qquad (P_n\neq 0)$$

defines the sequence $\{\bar{t}_n\}$ of $(\overline{N}, p_n)$-means of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series Σa_n is said to be *absolutely summable* $(\overline{N}, p_n)$, or *summable* $|\overline{N}, p_n|$, if

$$\Sigma|\bar{t}_n-\bar{t}_{n-1}|<\infty.$$

In the special case when $p_n=1$, for $n=0,1,2,\cdots$, $(\overline{N}, p_n)$-mean reduces to $(C,1)$-mean and then $|\overline{N}, p_n|$-summability is the same as summability $|C, 1|$.

It is known that*, in the special case when $p_n=(n+1)^{-1}$, $|\overline{N}, p_n|$ is equivalent to the summability $|R, \log n, 1|$.

2. Given a sequence $\{a_n\}$ we write $\Delta a_n=a_n-a_{n+1}$, $\Delta^m a_n=\Delta(\Delta^{m-1}a_n)$ with $\Delta^0 a_n=a_n$, where m is a positive integer. The sequence $\{a_n\}$ is said to be *convex* if $\Delta^2 a_n\geq 0$. It is well known that if $\{a_n\}$ is bounded and convex, then

*Mazhar (1966). This can be easily proved by employing the method used by Iyer (1963).

$$a_n \downarrow, \ n\Delta a_n \longrightarrow 0, \ n \longrightarrow \infty \text{ and } \sum_{n=1}^{\infty}(n+1)\Delta^2 a_n < \infty.$$

A sequence $\{a_n\}$ is said to be *quasi-convex* if

$$\sum_{n=1}^{\infty}(n+1)|\Delta^2 a_n| < \infty.$$

It is clear from the above result that every bounded convex sequence is quasi-convex. However, the converse need not be true. Contrary to what we have for convex sequences, a null quasi-convex sequence $\{a_n\}$ need not be monotonic decreasing. It is, however, of bounded variation and it satisfies the condition:

$$n\Delta a_n \longrightarrow 0, \ n \longrightarrow \infty.$$

The concept of quasi-convex sequence was recently generalized by Telyakovskii (1973). He defined the class S of sequences as follows:

A sequence $\{a_n\}$ is said to *belong to class S* if the following conditions are satisfied

(i) $a_n \longrightarrow 0, \ n \longrightarrow \infty,$

(ii) there exists a sequence of numbers $\{A_k\}$ such that $A_k \downarrow 0$ and $\sum_{k=1}^{\infty} A_k < \infty,$

(iii) $|\Delta a_k| \leq A_k$ for all k.

Taking $A_k = \sum_{m=k}^{\infty} |\Delta^2 a_m|$, it follows that a null quasi-convex sequence $\{a_n\}$ belongs to the class S. The converse is obviously not true. In view of the conditions (ii) and (iii), it follows that every sequence $\{a_n\}$ of class S is of bounded variation and that $n\Delta a_n \longrightarrow 0$, as $n \longrightarrow \infty$.

Now we give the concept of δ-quasi-monotonicity. A sequence $\{b_n\}$ of positive numbers is said to be *quasi-monotone* if $\Delta b_n \geq -\alpha b_n/_n$ for some positive α. It is obvious that every null monotonic decreasing sequnce is quasi-monotone. The sequence $\{b_n\}$ is said to be *δ-quasi-monotone* if $b_n \longrightarrow 0$, $b_n > 0$ ultimately and $\Delta b_n \geq -\delta_n$, where $\{\delta_n\}$ is a sequence of positive numbers. It is easy to see that a null quasi-monotone sequence is δ-quasi-monotone with $\delta_n = \alpha b_n/_n$.

A sequence $\{a_n\}$ will be said to *belong to class* $S(\delta)$ if

(i) $a_n \longrightarrow 0, \ n \longrightarrow \infty,$

(ii) there exists a sequence of numbers $\{A_n\}$ such that it is δ-quasi-monotone and $\sum_{n=1}^{\infty} A_n$ is convergent,

(iii) $|\Delta a_n| \leq |A_n|$ for all n.

It is trivial that $a_n \in S \Longrightarrow a_n \in S(\delta)$.

3. The concepts of convex and quasi-convex sequences have been applied to various types of problems in the theory of summability by several authors. Few of them are given below.

Mazhar (1966) proved the following theorem which is a generalization of a result of Bhatt (1960, 1962) on the $|R, \log n, 1|$-summability factors of infinite series.

THEOREM A (Mazhar (1966), Theorem 1). *If $\{\lambda_n\}$ is a convex sequence such that $\Sigma p_n \lambda_n < \infty$, where $\{p_n\}$ is non-increasing, and the sequence $\{\mu_n\}$, the $(\overline{N}, p_n)$ -mean of $\{a_n P_n / p_n\}$, satisfies the condition:*

$$|\mu_n| = 0(\gamma_n)(C, 1)^*,$$

$\{\gamma_n\}$ being a positive non-decreasing sequence such that $\Delta^2 \frac{1}{\gamma_n} \geq 0$, and $\Delta\gamma_n = 0\{p_n \gamma_n / P_n\}$, then the series $\Sigma a_n \lambda_n P_n / \gamma_n$ is summable $|\overline{N}, p_n|$.

In the special case when $p_n = 1$, for all n, this result also generalizes the following theorem of Prasad and Bhatt (1957).

THEOREM B (Prasad and Bhatt (1957, Theorem 1, with $\alpha = 1$)). *If $\{\lambda_n\}$ is a convex sequence such that $\Sigma n^{-1} \lambda_n < \infty$, and if the sequence $\{t_n^1\}$, the $(C, 1)$-mean of $\{na_n\}$, satisfies the condition:*

$$|t_n^1| = 0\{(\log n \mp 1)\}^k \ (C, 1), \ k \geq 0,$$

then the series $\Sigma \ \{\log(n+1)\}^{-k} \lambda_n a_n$ is summable $|C, 1|$.

Recently, Ahmad (1974) proved the following theorem, which is a further generalization of Theorem A.

THEOREM C ([2]). *Let the sequence $\{p_n\}$ be such that*

(ia) $p_{n+1}/P_{n+1} = 0 \ (p_n / P_n)$,

(ib) $\sum_{m=0}^{n} m |\Delta p_m| = 0(P_n)$,

(ic) $(n+1) p_n = 0(P_n)$.

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*This means that $\frac{1}{n+1} \sum_{\nu=0}^{n} |\mu_\nu| = 0(\gamma_n)$, as $n \to \infty$.

If the sequence $\{\varepsilon_n\}$ *is such that*

(iia) $P_n\varepsilon_n=0(1)$,

(iib) $\Sigma p_n|\varepsilon_n|<\infty$,

(iic) $\Sigma n P_n|\Delta^2\varepsilon_n|<\infty$,

the sequence $\{\mu_n\}$, *the* $(\overline{N}, p_n)$*-mean sequence of* $\{a_nP_n/p_n\}$, *satisfies the condition:*

$$|\mu_n|=0(r_n)\ (C,1),$$

$\{\gamma_n\}$ *being a positive non-decreasing sequence, and if* $\{\theta_n\}$, *be a positive sequence such that*

(iiia) $\gamma_n/\theta_n=0(P_n)$,

(iiib) $\Delta\theta_n=0\{p_n\theta_n/P_n\}$,

then the series $\Sigma a_n\varepsilon_n/\theta_n$ *is summable* $|\overline{N}, p_n|$.

The main object of the present paper is to give a further generalization of Theorem C, so that the previous theorems become special cases of our main theorem.

4. We assert the following main theorem

THEOREM. *Let the sequence* $\{p_n\}$ *be such that*

(a) $p_{n+1}/P_{n+1}=0(p_n/P_n)$,

(b) $\sum_{m=0}^{n} m|\Delta p_m|=0(P_n)$,

(c) $(n+1)p_n=0(P_n)$.

If the sequence $\{\varepsilon_n\}$ *is such that*

(a′) $\varepsilon_n=0(1)$,

(b′) *exists a sequence of numbers* $\{A_k\}$ *such that it is* δ*-quasi-monotone with:*

(i) $\sum_{n=1}^{\infty} n P_n\delta_n<\infty$,

(ii) $\Sigma P_{n+1}A_n<\infty$,

and

(c′) $|\Delta\varepsilon_n|\leq|A_n|$ *for all* n,

the sequence $\{\mu_n\}$, *the* $(\overline{N}, p_n)$*-mean sequence of* $\{a_nP_n/p_n\}$, *satisfies the condition:*

$$|\mu_n|=0(\gamma_n)\ (C,1),$$

$\{\gamma_n\}$ being a positive non-decreasing sequence, and if $\{\theta_n\}$ be a positive sequence such that

(a″) $\gamma_n/\theta_n = 0(P_n)$,

(b″) $\Delta\theta_n = 0\{p_n\theta_n/P_n\}$,

then the series $\sum_{n=1}^{\infty} a_n\varepsilon_n/\theta_n$ is summable $|\overline{N}, p_n|$.

It is to be noted that whenever we take

$$A_m = \sum_{n=m}^{\infty} P_n|\Delta^2\varepsilon_n|,\ P_n\varepsilon_n = 0(1),\ \Sigma p_n|\varepsilon_n| < \infty, \text{ and } \Sigma nP_n|\Delta^2\varepsilon_n| < \infty,$$

the conditions (a′), (b′) and (c′) of our theorem are automatically satisfied and hence Theorem C becomes a particular case of our theorem.

5. The proof of our theorem needs the following lemmas

LEMMA 1 (Robertson (1968), Theorems 1 and 2). *Let the sequence $\{a_n\}$ be δ-quasi-monotone such that $\Sigma\phi_n\delta_n < \infty$, $\{\phi_n\}$ being a positive monotonic increasing sequence. If $\Sigma a_n\Delta\phi_n < \infty$, then*

(i) $a_n\phi_n = 0(1)$, *as* $n \to \infty$,

(ii) $\sum_{n=0}^{\infty} \phi_{n+1}|\Delta a_n| < \infty$.

LEMMA 2. *Let the sequence $\{p_n\}$ be such that $(n+1)p_n = 0P(_n)$. If $\{A_n\}$ be a δ-quasi-monotone sequence with $\Sigma nP_n\delta_n < \infty$ and $\Sigma P_{n+1}A_n < \infty$, then*

(i) $n\,P_n\,A_n = 0(1)$, $n \to \infty$

and

(ii) $\Sigma(n+1)P_{n+1}|\Delta A_n| < \infty$.

PROOF. Taking $a_n = A_n$ and $\phi_n = nP_n$ in Lemma 1, we see that

$$\begin{aligned}\Sigma a_n\Delta\phi_n &= \Sigma A_n\Delta\phi_n \\ &= \Sigma\ \ _n\Delta(n\,P_n) \\ &= -\Sigma A_n(P_n + (n+1)p_{n+1}) \\ &\le \Sigma A_nP_n + \Sigma P_{n+1}A_n \\ &\le K\Sigma P_{n+1}A_n \\ &< \infty.\end{aligned}$$

Which means that the hypothesis of Lemma 1, is satisfied. Hence the result of this lemma follows directly by the use of Lemma 1.

LEMMA 3. *Under the hypothesis of the theorem, we have*

(i) $\Sigma \frac{p_\nu}{P_\nu} \frac{|\varepsilon_\nu|}{\theta_\nu} |\mu_\nu| < \infty$,

(ii) $\Sigma \frac{p_\nu}{P_{\nu+1}} \frac{|\varepsilon_{\nu+1}|}{\theta_{\nu+1}} |\mu_\nu| < \infty$.

PROOF OF (i). Following the proof of Lemma 3 of Ahmad ([2]), as $m \to \infty$, we obtain

$$\sum_{\nu=0}^{m} \frac{p_\nu}{P_\nu} \frac{|\varepsilon_\nu|}{\theta_\nu} |\mu_\nu|$$
$$\leq K \sum_{\nu=0}^{m-1} P_\nu |\Delta\varepsilon_\nu| + K \sum_{\nu=0}^{m-1} p_{\nu+1} |\varepsilon_{\nu+1}| + K P_m |\varepsilon_m|,$$
$$= K(J_1^m + J_2^m + J_3^m), \text{ say}$$

by hypothesis. Therefore, in order to prove the validity of (i) we have to show that, as m→∞,

$$J_r^m \leq K, \text{ for } (r=1, 2, 3) \tag{5.1}$$

PROOF OF (5.1). As $m \to \infty$,

$$J_1^m = \sum_{\nu=0}^{m-1} P_\nu |\Delta\varepsilon_\nu|$$
$$\leq \sum_{\nu=0}^{m-1} P_\nu |A_\nu|$$
$$< \infty,$$

by hypothesis;

$$J_2^m = \sum_{\nu=0}^{m-1} p_{\nu+1} |\varepsilon_{\nu+1}|$$
$$= \sum_{\nu=0}^{\infty} p_{\nu+1} \left|\sum_{n=\nu}^{\infty} \Delta\varepsilon_{n+1}\right|$$
$$\leq \sum_{\nu=0}^{\infty} p_{\nu+1} \sum_{n=\nu}^{\infty} |\Delta\varepsilon_{n+1}|$$
$$= \sum_{n=0}^{\infty} |\Delta\varepsilon_{n+1}| \sum_{\nu=0}^{n} p_{\nu+1}$$
$$= \sum_{n=0}^{\infty} P_{n+1} |\Delta\varepsilon_{n+1}|$$

$$\leq \sum_{n=0}^{\infty} P_{n+1}|A_{n+1}|$$
$$<\infty,$$

by hypothesis;

$$J_3^m = P_m|\varepsilon_m|$$
$$= P_m|\sum_{\nu=m}^{\infty}\Delta\varepsilon_\nu|$$
$$\leq P_m\sum_{\nu=m}^{\infty}|\Delta\varepsilon_\nu|$$
$$\leq P_m\sum_{\nu=m}^{\infty}|A_\nu|$$
$$\leq \sum_{\nu=m}^{\infty}P_\nu|A_\nu|$$
$$<\infty,$$

by hypothesis.

This completes the proof of Lemma 3(i).

The proof of the inequality (ii) is similar.

6. Proof of the theorem

Now we are ready to give the proof of the main theorem.

Let $\{\overline{\tau}_n\}$ be the sequence of $(\overline{N}, p_n)$-mean of the series $\Sigma a_n\varepsilon_n/\theta_n$. Then, by definition, we have

$$\overline{\tau}_n - \overline{\tau}_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n} P_{\nu-1}\, a_\nu\, \varepsilon_\nu/\theta_\nu$$
$$= \frac{p_n}{P_n P_{n-1}}\Sigma, \text{ say.}$$

where,

$$\Sigma = \sum_{\nu=0}^{n}\frac{P_{\nu-1}}{P_\nu}P_\nu a_\nu\varepsilon_\nu/\theta_\nu$$
$$= -\sum_{\nu=0}^{n-1}\frac{p_\nu}{P_\nu}\frac{\varepsilon_\nu}{\theta_\nu}P_\nu\mu_\nu + \sum_{\nu=0}^{n-1}\frac{P_\nu}{p_\nu}\frac{p_{\nu+1}}{P_{\nu+1}}\frac{\varepsilon_\nu}{\theta_\nu}P_\nu\mu_\nu$$
$$+\sum_{\nu=0}^{n-1}\frac{P_\nu}{P_{\nu+1}}\frac{\Delta\varepsilon_\nu}{\theta_\nu}P_\nu\mu_\nu - \sum_{\nu=0}^{n-1}\frac{P_\nu P_\nu}{P_{\nu+1}}\frac{\Delta\theta_\nu}{\theta_\nu\theta_{\nu+1}}\varepsilon_{\nu+1}$$
$$+\frac{P_{n-1}}{P_n}\frac{\varepsilon_n}{\theta_n}P_n\mu_n$$

$$=\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5, \text{ say.}$$

Hence, in order to prove the theorem, we have to show that

$$\Sigma\frac{p_n}{P_nP_{n-1}}|\Sigma_r|\leq K\ (r=1,\ 2,\ \cdots,\ 5). \tag{6.1}$$

PROOF OF(6.1). For $\nu\geq 0$, we have

$$\sum_{n=\nu+1}^{\infty}\frac{p_n}{P_nP_{n-1}}\leq K/P_\nu.$$

Now, we have

$$\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}|\Sigma_1|\leq\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}\sum_{\nu=0}^{n-1}p_\nu\frac{|\varepsilon_\nu|}{\theta_\nu}|\mu_\nu|$$

$$=\sum_{\nu=0}^{\infty}p_\nu\frac{|\varepsilon_\nu|}{\theta_\nu}|\mu_\nu|\sum_{n=\nu+1}^{\infty}\frac{p_n}{P_nP_{n-1}}$$

$$\leq K\sum_{\nu=0}^{\infty}\frac{p_\nu}{P_\nu}\ \frac{|\varepsilon_\nu|}{\theta_\nu}|\mu_\nu|$$

$$\leq K,$$

by hypothesis and Lemma 3(i);

$$\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}|\Sigma_2|\leq\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}\sum_{\nu=0}^{n-1}\frac{p_{\nu+1}}{P_{\nu+1}}\ \frac{|\varepsilon_\nu|}{\theta_\nu}|\mu_\nu|P_\nu$$

$$=\sum_{\nu=0}^{\infty}\frac{p_{\nu+1}}{P_{\nu+1}}\ \frac{|\varepsilon_\nu|}{\theta_\nu}|\mu_\nu|P_\nu\sum_{n=\nu+1}^{\infty}\frac{p_n}{P_nP_{n-1}}$$

$$\leq K\sum_{\nu=0}^{\infty}\frac{p_{\nu+1}}{P_{\nu+1}}\ \frac{|\varepsilon_\nu|}{\theta_\nu}|\mu_\nu|$$

$$\leq K\sum_{\nu=0}^{\infty}\frac{p_\nu}{P_\nu}\ \frac{|\varepsilon_\nu|}{\theta_\nu}|\mu_\nu|$$

$$\leq K$$

$$\left(\text{since } \frac{p_{\nu+1}}{P_{\nu+1}}=0\left(\frac{p_\nu}{P_\nu}\right)\right)$$

by hypothesis and Lemma 3(i);

$$\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}|\Sigma_3|\leq\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}\sum_{\nu=0}^{n-1}P_\nu\frac{|\Delta\varepsilon_\nu|}{\theta_\nu}|\mu_\nu|$$

$$=\sum_{\nu=0}^{\infty}P_\nu\frac{|\Delta\varepsilon_\nu|}{\theta_\nu}|\mu_\nu|\sum_{n=\nu+1}^{\infty}\frac{p_n}{P_nP_{n-1}}$$

$$\leq K\sum_{\nu=0}^{\infty}\frac{|\Delta\varepsilon_\nu|}{\theta_\nu}|\mu_\nu|$$

$$\leq K\sum_{\nu=0}^{m}\frac{|A_\nu|}{\theta_\nu}|\mu_\nu|,\quad (m\to\infty)$$

$$=K\Big[\sum_{\nu=0}^{m-1}\frac{|\Delta A_\nu|}{\theta_\nu}\nu_\nu(\nu+1)+\sum_{\nu=0}^{m-1}\frac{|\Delta\theta_\nu|}{\theta_\nu\theta_{\nu+1}}A_{\nu+1}\nu_\nu(\nu+1)$$

$$+(m+1)\frac{\nu_m}{\theta_m}A_m\Big],\quad (m\to\infty)$$

$$\leq K\Big[\sum_{\nu=0}^{m}\nu P_\nu|\Delta A_\nu|+\sum_{\nu=0}^{m-1}P_{\nu+1}|A_{\nu+1}|+m\,P_m\,A_m\Big],$$

$$(m\to\infty)$$

$$\leq K,$$

by hypothesis and Lemma 2;

$$\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}|\Sigma_4|\leq\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}\sum_{\nu=0}^{n-1}\frac{P_\nu^2}{P_{\nu+1}}\ \frac{|\Delta\theta_\nu|}{\theta_\nu\theta_{\nu+1}}|\varepsilon_{\nu+1}||\mu_\nu|$$

$$\leq K\sum_{\nu=0}^{\infty}\frac{p_\nu}{P_{\nu+1}}\ \frac{|\varepsilon_{\nu+1}|}{\theta_{\nu+1}}P_\nu|\mu_\nu|\sum_{n=\nu+1}^{\infty}\frac{p_n}{P_nP_{n-1}}$$

$$\leq K\sum_{\nu=0}^{\infty}\frac{p_\nu}{P_{\nu+1}}\ \frac{|\varepsilon_{\nu+1}|}{\theta_{\nu+1}}|\mu_\nu|$$

$$\leq K,$$

by hypothesis and Lemma 3(ii); and

$$\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}|\Sigma_5|=\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}\ \frac{P_{n-1}}{P_n}\ \frac{|\varepsilon_n|}{\theta_n}P_n|\mu_n|$$

$$=\sum_{n=1}^{\infty}\frac{p_n}{P_n}\ \frac{|\varepsilon_n|}{\theta_n}|\mu_n|$$

$$\leq K,$$

by hypothesis and Lemma 3(i).

This completes the proof of the theorem.

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