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ON OSCILLATION OF SECOND ORDER DIFFERENTIAL EQUATIONS

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1. Introduction

In this paper, we shall study the oscillation and stability of the nonlinear differential equation

$$(a(t)\ x'(t))'+b(x)=0, \tag{1}$$

and the corresponding pertured equation

$$(a(t)\ x'(t))'+b(x)=f(t,\ x,\ x). \tag{2}$$

In articles [2—9], the behaviour of the oscillatory, nonoscillatory solutions of some equations of second order have been discussed. Relatively few oscillations criteria are known for equation with perturbations see [8], and some of the references therein.

DEFINITION 1. A solution $x(t)$ of (1) is said to be *stable* (in the sence of Liapunov) if the solution $x(t)$ and its derivative $x(t)$ are bounded, [1].

DEFINITION 2. A solution $x(t)$ of (1) or (2) is said to be *oscillatory* if there is an unbounded set of zeros of $x(t)$ i.e if $t_1\geqslant 0$, then there is a $t>t_1$ such that $x(t)=0$. If there is a $t_1>0$ such that $x(t)\neq 0$ for $t>t_1$, then the solution $x(t)$ is called *nonoscillatory*. Equation (1) or (2) is said to be *oscillatory* (or *B-oscillatory*) if all its solutions (bounded solutions) are oscillatory.

2. In this section we shall prove that the solutions of (1) are stable and oscillatory.

Our main assumptions are

(1) $a(t)\in C^1[0,\ \infty)$, $a'(t)\geqslant 0$, and $a(t)$ is bounded i.e $a_1\geqslant a(t)\geqslant a_0>0$, a_0, $a_1\in R$, $a'(t)\longrightarrow 0$ as $t\rightarrow\infty$.

(2) $xb(x)>0$ and $b'(x)\geqslant 0$ for $x\neq 0$.

(3) $B(x)=\int_0^x b(u)\ du$, $\lim_{|x|\rightarrow\infty} B(x)\longrightarrow\infty$.

We need the following Lemma which proved by Utz [11] in proving theorem 1.

LEMMA. *Suppose that $x(t)$ is a real function for which $x(t)$ is defined for $t\geqslant a$,*

(i) *If for all* $t \geqslant t_1 \geqslant a$, $x'(t) < 0$, $x''(t) \leqslant 0$, *then* $\lim_{t\to\infty} x(t) = -\infty$.

(ii) *If for all* $t \geqslant t_1 \geqslant a$, $x'(t) < 0$, $x''(t) \geqslant 0$, *then* $\lim_{t\to\infty} = +\infty$.

THEOREM 1. *Under assumptions* 1—3, *the soltuions of* (1) *are stable and B-oscillatory.*

PROOF. We shall prove, first, that the solutions of (1) are stable. Multiplying both sides of (1) by $a(t)x'(t)$ and integrating from 0 to t, we obtain:

$$\frac{1}{2}(a(t)\ x'(t))^2 + a(t)\ B(x(t)) - \int_0^t B(x(u))\ a'(u)due = k, \tag{3}$$

where $k = \frac{1}{2}(a(0)x'(0))^2 + a(0)\ B(x(0))$.

Thus

$$a(t)\ B(x(t)) \leqslant k + \int_0^t B\ (x(u))\ a(u)\,[a'(u)/a(u)]\ du.$$

Hence by Gronwall's inequality it follows that

$$a(t)\ B(x(t)) \leqslant ka(t)/a(0), \text{ i.e}$$

$$B(x(t)) \leqslant k/a(0). \tag{4}$$

Hence, by assumption (3), $x(t)$ must remains bounded as $t \to \infty$. It is clear that from (3) and (4), that $|x'|$ is bounded.

Thus we can say, according to definition 1, that the solutions of (1) are stable.

Secondly to prove that the solutions of (1) are B-oscillatory, we assume $x(t) > 0$. On the contrary, suppose $x(t)$ is a nonoscillatory solution of (1), then $x(t)$ and consequently $x'(t)$ must be of fixed sign. Otherwise we have the following cases:

(i) If $x(t) = 0$; we have, from (1):

$$x''(t) = -b(x)/a(t) < 0.$$

This means that the solution $x(t)$ has an infinite number of relative maxima. Thus this case is impossible.

(ii) If $x'(t) > 0$; Integrating (1) from 0 to t, we have

$$a(t)x'(t) = a(0)x'(0) - B(x(t)) \tag{5}$$

Hence, by assumption (3), the right hand side of (5) tends to $-\infty$ as $t \to \infty$. Thus this case is impossible.

(iii) If $x'(t) < 0$: By assumption (1) then as $t \to \infty$ it follows that

$$x''(t) = -\,[b(x(t)) + a'(t)x'(t)]\,/a(t) \leqslant 0.$$

Then by the above lemma, part (i), $\lim_{t\to\infty} x(t) = -\infty$.

Which is a contradiction. Thus $x(t)$ must be oscillatory solution of (1). By the first part of the theorem it follows that the solutions are B-oscillatory.

REMARK. The proof in case $x(t)<0$ is similar and has been omitted.

3. In this section we shall consider the perturbed equation (2), namely,

$$(a(t)\ x'(t))'+b(x)=f(t,\ x,\ x'). \tag{2}$$

We assume the following:

(i) There exist a continuous functions $u(x)$ such that $f(t,\ x,\ x')/u(x)<V(t)$.

(ii) $xu(x)>0,\ \ u'(x)\geqslant k>0\ \ x\neq 0$.

(iii) $a(t)\leqslant a_1$ where $a_1,\ k\in R$.

THEOREM 2. *Under the assumptions* i—iii *of this section, if*

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_1}^{t}\int_{t_1}^{r}(b(x(s))/u(x(s))-V(s))\ ds\ dr=\infty, \tag{6}$$

then all solutions of (2) *are oscillatory.*

PROOF. On the contrary, suppose that $x(t)$ does not oscillate. Thus $x(t)$ and consequently $x'(t)$ have fixed sign. Let $x(t)\neq 0$ for $t\geqslant t_2>t_1$.

Integrating (2) twice from t_2 to t we obtain:

$$\int_{t_2}^{t}[a(s)x'(s)/u(x(s))]ds+\frac{k}{a_1}\int_{t_2}^{t}\left(\frac{a(s)x'(s)}{u(x(s))}\right)^2 ds$$

$$+\int_{t_2}^{t}\int_{t_2}^{s}(b(x(s))/u(x(s))-V(s))\ ds\leqslant c_1 t,\ \ t\geqslant t_2.$$

It follows, from (6), that

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_2}^{t}\left(\frac{a(s)x'(s)}{u(x(s))}\right)ds=-\infty.$$

Let R denote $\left|\int_{t_2}^{t}\left(\frac{a(s)x'(s)}{u(x(s))}\right)ds\right|$, then

from Schwartz's inequality for integral, it follows that;

$$R^2(t)\leqslant\int_{t_2}^{t}\left[\frac{a(s)x'(s)}{u(x(s))}\right]^2 ds.$$

Hence

$$(K^2/a_1^2t^2)\left(\int_{t_1}^{t}\frac{R^2(s)}{s}\ ds\right)^2\leqslant\frac{R^2(t)}{t^2}.$$

Also let $p(t)=\int_{t_2}^{t}(R^2(s)/s)\,ds,\ t\geq t_2$.

Then

$$K^2/a_1^2 t\leq p'(t)p^2(t),\quad t\geq t_2$$

and from which it follows that

$$\frac{K^2}{a_1^2}\ \ln\left(\frac{t}{t_2}\right)\leq\frac{1}{p(t_2)}-\frac{1}{p(t)}\leq 1/p(t_2).$$

Which is a contradiction. This completes the proof.

REMARK. The proof in case $x(t)<0$ for $t\geq t_2$ is similar and has been omitted.

GENERAL REMARK. Many research papers, for example see [1, 4, 10] have been published when the perturbing function is small, i.e the equation (2) has the form,

$$[a(t)\ x'(t)]'+b(x)=\mu f(t,\ x,\ x',\ \mu),$$

where μ is a small parameter.

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