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ZEROS OF PARTIAL SUMS OF THE LAURENT SERIES OF ANALYTIC FUNCTIONS

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It is shown that there exists a sequence of zeros c_n of partial sums of the Laurent expansion around an isolated essential singularity of an analytic function f such that $\lim_n f(c_n)=0$ provided the Laurent expansion has an infinite nonprincipal part.

Let $\Sigma_{-\infty}^{\infty} a_m(z-a)^m$ be the Laurent series (around a) of an analytic function f and let $v<u$ be two integers. Then the function $\Sigma_v^u a_m(z-a)^m$ is called a *partial sum* of the corresponding Laurent series of f. If a is an isolated essential singularity of f then, as expected, c is called a *Picard exceptional value* of f if there exists a neighborhood D of a such that $f(z)\neq c$ for every $z\in D$. The well known Picard's great theorem states that f has at most one Picard exceptional value.

This paper proves a variant (see Theorem 2 below) of the following:

Abian's conjecture. *Let* 0 *be an isolated essential singularity of an analytic function f. Then there exists a sequence of complex numbers c_n converging to* 0 *such that every c_n is a zero of some partial sum T_n of the corresponding Laurent series of f around* 0 *and such that $\lim_n f(c_n)=0$.*

Abian's conjecture can be strengthened in the following two ways:

(i) by requiring that every c_n be a zero of some partial sum T_n which belongs to a preassigned sequence (T_n) of partial sums which converges to f in a deleted neighborhood of 0.

(ii) by requiring that the c_n's be zeros of all but a finite number of partial sums T_n which belong to a preassigned sequence (T_n) of partial sums which converges to f in a deleted neighborhood of 0.

In [1] the (ii) version of Abian's conjecture is proved provided "0 is not a

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Picard exceptional value of f" and in [2] the (i) version of Abian's conjecture is proved provided "0 is an essential singularity of f of finite order".

In Theorem 2 below, we prove Abian's conjecture without requiring either of the two abovementioned provisions. However, we require that the corresponding Laurent series has an infinite nonprincipal part.

Our proof of Theorem 2 is based on Theorem 1 which is proved in [1] and which is stated below (for the sake of completeness of the papar) without proof.

THEOREM 1. *Let 0 be an isolated essential singularity of an analytic function f such that 0 is not a Picard exceptional value of f. Let there be preassigned a sequence of partial sums T_n (of the corresponding Laurent series of f around 0) which converges to f in a deleted neighborhood of 0. Then there exists a sequence of complex numbers c_n converging to 0 and an integer $N>0$ such that $T_{N+n}(c_n)=0$ for every $n\in\omega$ and $\lim_n f(c_n)=0$.*

We also need the following:

LEMMA. *Let 0 be an isolated essential singularity of an analytic function whose corresponding Laurent series $\Sigma_{-\infty}^{\infty} a_m z^m$ has an infinite nonprincipal part (i.e., $a_m\neq 0$ for infinitely many positive m's). Let n be an integer. Then there exists $u\geq n$ such that 0 is not a Picard exceptional value of $\Sigma_{-\infty}^{u} a_m z^m$.*

PROOF. Assume on the contrary that for every $u\geq n$ it is the case that 0 is the Picard exceptional value of $\Sigma_{-\infty}^{u} a_m z^m$. From the hypothesis of the Lemma it follows that there exists the smallest integer $s>n$ such that $a_s\neq 0$. Clearly, from our assumption it follows that 0 is the Picard exceptional value of $\Sigma_{-\infty}^{n} a_m z^{m-s} =z^{-s}(\Sigma_{-\infty}^{n} a_m z^m)$. Hence 0 is not the Picard exceptional value of $a_s+\Sigma_{-\infty}^{n} a_m z^{m-s}$ which implies that 0 is not the Picard exceptional value of $\Sigma_{-\infty}^{s} a_m z^m=z^s(a_s+\Sigma_{-\infty}^{n} a_m z^{m-s})$ which contradicts our assumption since $s>n$. Thus, the Lemma is proved.

Based on Theorem 1 and the Lemma we prove:

THEOREM 2. *Let 0 be an isolated essential singularity of an analytic function f whose Laurent series $\Sigma_{-\infty}^{\infty} a_m z^m$ around 0 has an infinite nonprincipal part. Then there exist a sequence of complex numbers c_n converging to 0 and a sequence of partial sums $T_n(z)=\Sigma_{v_n}^{u_n} a_m z^m$ of the Laurent series such that $T_n(c_n)=0$ for every $n\in\omega$ and $\lim_n f(c_n)=0$.*

PROOF. Let n be a natural number (i.e., $n \in \omega$).

From the Lemma it follows that there exists $u_n > n$ such that 0 is not a Picard exceptional value of $\Sigma_{-\infty}^{u_n} a_m z^m$. Also the continuity of the power series $\Sigma_{u_n+1}^{\infty} a_m z^m$ implies that there exists a disk D_n with center at 0 and with radius $< 2^{-n}$ such that

(1) $$|\Sigma_{u_n+1}^{\infty} a_m z^m| < 2^{-n-1} \text{ for every } z \in D_n$$

Now, applying Theorem 1 to the function $\Sigma_{-\infty}^{u_n} a_m z^m$ and to the sequence of partial sums $T_k(z) = \Sigma_{-k}^{u_n} a_m z^m$ we see that there exist $v_n < -n$ and a complex number c_n such that

(2) $$\Sigma_{v_n}^{u_n} a_m c_n^m = T_n(c_n) = 0 \text{ with } c_n \in D_n$$

and

(3) $$|\Sigma_{-\infty}^{u_n} a_m c_n^m| < 2^{-n-1}$$

Clearly, (1) and (3) imply:

(4) $$|\Sigma_{-\infty}^{\infty} a_m c_n^m| = |f(c_n)| < 2^{-n}$$

But then, obviously, (2) and (4) imply the conclusion of the Theorem.

REMARK. In view of Theorem 2, in order to settle Abian's conjecture (with the possibility of strengthening it by (i) or (ii)) affirmatively, it remains to verify it only for the case of entire transcendental functions (in fact, in view of [2] and [1], only for those of infinite order and having 0 as their Picard exceptional value), i.e.,

Abian's conjecture. *Let f be an entire transcendental function. Then there exists an unbounded sequence of complex numbers c_n such that every c_n is a zero of some partial sum of the Taylor series (around 0) of f and such that* $\lim_n f(c_n) = 0$.

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