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PERFECT MAPPINGS AND SINGULAR SETS

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1. A mapping (continuous function) $f: X \to Y$ is a *compact mapping* if $f^{-1}(K)$ is compact for each compact $K \subset Y$. The mapping is *perfect* if f is closed and $f^{-1}(y)$ is compact for each $y \in Y$. If Y is a k-space the two concepts are the same. A set of points in Y where f fails to be perfect is called a *singular set* of the mapping f. Whyburn (4,5) and Cain (2) defined singular sets in their studies of mappings where, generally, the spaces X and Y were locally compact separable metric spaces. In that setting their singular sets coincide but in less restrictive spaces they are not usually equivalent. The purpose of this paper is to study these and other singular sets of the mapping f. We assume throughout that X and Y are at least Hausdorff spaces and Y a k-space. Any other condition on the spaces will be mentioned explicitly.

2. Whyburn's concept of a Singular set.

DEFINITION 2.1. Let $f: X \to Y$ be a mapping and let $B \subset Y$. Any set $A \subset X$ such that $f(A)=B$ is called a *trace* of B. If A is a compact set, then we say that B has *compact trace.*

DEFINITION 2.2. Let $f: X \to Y$ be a mapping and let Q_1 be the union of the interiors of all sets K such that $f^{-1}(K)$ is compact. The set $S_1=Y-Q_1$ is called a *singular set* for the mapping f.

DEFINITION 2.3. Let $f: X \to Y$ be a mapping and let Q_2 be the union of the interiors of all sets having a compact trace. Let $S_2=Y-Q_2$.

THEOREM 2.1. *The sets S_1 and S_2 are closed sets.*

PROOF. Q_1 and Q_2 are unions of interiors of certain sets and thus are open sets. Their respective complements, S_1 and S_2 are closed sets.

The following theorem was established by E. A. Michael (2, Cor. 2.1).

THEOREM 2.2. *If $f: X \to Y$ is a closed mapping of the paracompact space X onto Y, then every compact subset of Y is the image of a compact set in X.*

With this result the following can be proved.

THEOREM 2.3. *Let $f: X \to Y$ be a closed mapping of the paracompact space X onto a locally compact space Y. Then each point in Y is interior to a set with compact trace, i.e., $S_2 = \phi$.*

PROOF. Let $y \in Y$. Since Y is locally compact, y has a neighborhood U such that cl$U$ is compact. By Theorem 2.2, cl$U$ has a compact trace.

The following theorem is a generalization of a theorem of Whyburn (4, Thm. 2.1). In that work he assumed X and Y were locally compact separable metric spaces, and the metric properties were essential in his proof.

THEOREM 2.4. *Let $f: X \to Y$ be a mapping. If each point of Y is interior to the image of some compact set, then each compact set in Y has compact trace.*

PROOF. Let K be a compact set in Y. For each $y \in K$ there exists a compact set $C(y) \subset X$ such that $y \in \mathrm{Int} f(C(y))$. Thus, $K \subset \bigcup \{\mathrm{Int}(C(y)) : y \in K\}$. Since K is compact, there is a finite family $\{\mathrm{Int} f(C(y_i)) : 1 \leq i \leq n\}$ such that $K \subset \bigcup \{\mathrm{Int} f(C(y_i)) : 1 \leq i \leq n\}$. Now let $C = [\bigcup \{C(y_i) : 1 \leq i \leq n\}] \cap f^{-1}(K)$. Then C is a compact set and $f(C) = K$.

THEOREM 2.5. *Let $f: X \to Y$ be a mapping where Y is a locally compact space. If each compact set in Y has a compact trace, then each point in Y is interior to the image of some compact set.*

PROOF. For $y \in Y$, let U be a neighborhood of y such that cl$U$ is compact. Now, there exists a compact $C \subset X$ such that $f(C) = \mathrm{cl}U$. Hence, $y \in \mathrm{Int} f(C)$.

In 1966, Whyburn (5) showed that if $f: X \to Y$ is a monotone mapping ($f^{-1}(y)$ is a continuum for each $y \in Y$) and X and Y are locally compact Hausdorff spaces, then $S_1 = S_2$. The following example shows that, in general, $S_1 \neq S_2$.

EXAMPLE 2.1. Let X be the set of real numbers with the usual topology and let $Y = [-1, 1]$ be a subspace of X. Define the mapping $f: X \to Y$ such that

$$f(x) = x \text{ for } -1 \leq x \leq 1,$$
$$f(x) = 1 \text{ for } x > 1,$$
$$f(x) = -1 \text{ for } x < -1.$$

For the compact set $K = [-1, 1] \subset X$, $f(K) = Y$. Thus, $S_2 = \phi$. However, for any compact neighborhood L of -1 or 1, $f^{-1}(L)$ is not compact. Hence, $S_1 \neq \phi$. Note that for $x' \in Y - \{-1, 1\}$ if $\varepsilon = \frac{1}{2}[\min\{|x'-1|, |x+1|\}]$, then $x \in [x'-\varepsilon, x'+\varepsilon]$ which is compact in Y and $f^{-1}[x'-\varepsilon, x'+\varepsilon] = [x'-\varepsilon, x'+\varepsilon]$, a compact set. Thus, $S_1 = \{-1, 1\}$.

THEOREM 2.6. *For any mapping* $f: X \to Y$, $S_2 \subset S_1$.

Proof. Since any set with compact inverse image has a compact trace, then $Q_1 \subset Q_2$. Hence, by De Morgan's laws, $S_2 \subset S_1$.

We include the proof of the following known result since the ideas used are needed in the proof of the next theorem.

LEMMA. *Let* $f: X \to Y$ *be a closed mapping,* H *an open set in* X *and let*

$$H_0 = \cup \{f^{-1}(y) : f^{-1}(y) \subset H\}.$$

Then H_0 *is an open set in* X.

PROOF. Since f is a closed mapping and $H \subset X$ is an open set, then $f(X-H)$ is closed. Now, $f[H_0 \cup (X-H)] = Y$ and $f(H_0) \cap f(X-H) = \phi$.
Thus, $f(H_0) = Y - f(X-H)$ is open in Y and therefore

$$H_0 = f^{-1}(f(H_0)) \text{ is open in } X.$$

THEOREM 2.7. *If* $f: X \to Y$ *is a mapping,* Y *a regular space and* y *a point of* $Y - S_1$, *then for any neighborhood* U *of* $f^{-1}(y)$, *there exists a neighborhood* V *of* y *such that* $f^{-1}(V) \subset U$.

PROOF. Let $y \in Y - S_1$ and let U be any neighborhood of $f^{-1}(y)$. Since Y is regular and $Y - S_1$ is open, there exists a neighborhood V_0 of y such that $\mathrm{cl}V_0 \subset Y - S_1$ and such that $f^{-1}(\mathrm{cl}V_0)$ is compact. If $f^{-1}(V_0) \subset U$, then the conclusion of the theorem is satisfied.

Assume $f^{-1}(V_0) \not\subset U$. The mapping $f | f^{-1}(\mathrm{cl}V_0)$ is a closed mapping. Let $H_0 = \cup \{f^{-1}(z) : f^{-1}(z) \subset f^{-1}(V_0) \cap U\}$. Then H_0 is open and $f(H_0)$ is open in V_0 and therefore open in Y. Then $V = f(H_0)$ satisfies the conclusion of the theorem.

THEOREM 2.8. *Under the conditions of Theorem 2.7, let* $Y_0 = Y - S_1$.

Then $g=f|f^{-1}(Y_0)$ *is a perfect mapping.*

PROOF. The results of the previous theorem show that g is a closed mapping and for $y\in Y-S_1$, $f^{-1}(y)$ is a compact set. Thus, g is a perfect mapping.

The set Y_0 is an open subset of Y. Cain (2, Cor. 3.9) erroneously concluded that under somewhat more restricted conditions such an open set would be dense in Y. He asserted that for a closed mapping $f: X\to Y$ where f is onto and X, Y are locally compact separable metric spaces, there is an open dense set $G\subset Y$ such that $f|f^{-1}(G)$ is a compact mapping.

While it is certainly posssible to have such a dense open subset, the conditions given do not insure its existence as the following example shows.

EXAMPLE 2.2 Let $X=[0,1)\cup(2,3]$ be a space with the relative topology from the real line and $Y=\{a, b\}$ with the discrete topology. Define $f: X\to Y$ be

$$f(x)=a \quad \text{for} \quad x\in[0,1),$$
$$f(x)=b \quad \text{for} \quad x\in(2,3].$$

Then f is closed and continuous but there is no dense open subset $G\subset Y$ such that $f|f^{-1}(G)$ is a compact mapping.

3. Cain's concept of a Singular Set.

DEFINITION 3.1. Let $f: X\to Y$ be a mapping. Let S_3 be the set of all points $y\in Y$ such that each neighborhood of y contains a compact set K such that $f^{-1}(K)$ is not compact.

This definition of a singular set for a mapping is due to G. L. Cain, Jr. (2), who studied certain mappings by investigating the properties of the set S_3 of singular points and its inverse image.

In this section some properties of the set S_3 are investigated and a comparison is made between the set S_3 and the S_1 of Section 2.

THEOREM 3.1. *S_3 is a closed set.*

PROOF. Let $y\in \mathrm{cl}S$. For any neighborhood U of y, $U\cap S\neq\phi$. For each $y'\in U\cap S_3$, U is a neighborhood of y' and as such, U contains a compact set K such that $f^{-1}(K)$ is not compact. Therefore, $y\in S_3$ and S_3 is a closed set.

THEOREM 3.2. $S_3\subset S_1$.

PROOF. It will be shown that $Y-S_1 \subset Y-S_3$.

Let $y \in Y-S_1$. Then there exists a neighborhood U of y such that $f^{-1}(\mathrm{cl}U)$ is a compact set. Let C be any compact set in U. Then $f^{-1}(C) \subset f^{-1}(\mathrm{cl}U)$ and hence $f^{-1}(C)$ is a compact set. This implies $y \in Y-S_3$.

The following example shows that, in general, $S_1 \neq S_3$.

EXAMPLE 3.1. Let $f: X \to Y$ be the identity mapping where both X and Y are the set of rational numbers with the subspace topology from R, the space of real numbers with the usual topology. Clearly, $f^{-1}(K)$ is compact for each compact set $K \subset Y$. Hence, $S_3 = \phi$. However, the interior of each compact set in Y is empty. Therefore, $S_1 = Y$.

THEOREM 3.3. *If $f: X \to Y$ is a mapping and Y is a locally compact Housdorff space, then $S_1 = S_3$.*

PROOF. If f is a perfect mapping, $S_1 = \phi = S_3$. If f is not perfect, by Theorem 3.2, we need only show $S_1 \subset S_3$.

Let $y \in Y-S_3$. Then there exists some neighborhood U of y such that for each compact set $K \subset U$, $f^{-1}(K)$ is compact. Let W be a neighborhood of y such that $\mathrm{cl}W$ is compact. There exists a neighborhood V of y such that $y \in \mathrm{cl}V \subset U$ and

$$y \in V \cap W \subset \mathrm{cl}(V \cap W) \subset U.$$

Now, $\mathrm{cl}(V \cap W)$ is a compact set and since $\mathrm{cl}(V \cap W) \subset U$, then $f^{-1}(\mathrm{cl}(V \cap W))$ is compact. Hence, y is interior to the set $\mathrm{cl}(V \cap W)$ whose inverse image is compact and, thus, by definition, $y \in Y-S_1$. Therefore, $Y-S_3 \subset Y-S_1$, or equivalently, $S_1 \subset S_3$.

In his papers, Cain (1, Thm. 2.4) states the following.

THEOREM 3.4. *If Y is locally compact, then for any point $y \notin S_3$ and any neighborhood U of $f^{-1}(y)$, there is a neighborhood W of y such that $f^{-1}(W) \subset U$.*

Some exception may be taken to the proof as given. However, the result now follows from Theorem 3.3 and Theorem 3.7.

While Theorem 3.3 shows that local compactness of Y is a sufficient condition for the equality of S_3 and S_1, the following example shows that local compactness is not necessary for their equality.

EXAMPLE 3.2. Let $X = \{-1\} \cup \{x : 0 < x < 1\} \subset R$. Let

$$Y=\{(0,0)\}\cup\{(x,y): y=\sin\frac{1}{x},\ 0<x<1\}$$

be a subspace of the plane. Y is not locally compact. Define $f: X\to Y$ such that f maps $X-\{-1\}$ homeomorphically onto $Y-\{(0,0)\}$ and $f(-1)=(0,0)$. f is clearly a continuous function. Each $y\in Y-\{(0,0)\}$ is interior to a closed segment of the curve $y=\sin\frac{1}{x}$, $0<x<1$, whose inverse image is compact. However, $(0,0)$ is not interior to any compact set. Thus, $S=\{(0,0)\}$. Now for any neighborhood U of $(0,0)$, there exists a number $N>0$ such that for any integer $n>N$, the set

$$K=\{(0,0)\}\ \cup\left\{\left(\frac{1}{n\pi},\ 0\right): n>N\right\}\subset U.$$

The set K is compact. However,

$$f^{-1}(K)=f^{-1}(0,0)\cup\left\{f^{-1}\left(\frac{1}{n\pi},\ 0\right): n>N\right\}$$

compact in X. Hence, $(0,0)\in S_3$. Therefore, $S_1=S_3$.

4. More on singular sets.

A mapping $f: X\to Y$ may fail to be perfect because for some $y\in Y$, $f^{-1}(y)$ is not compact or because the image of some closed set in X fails to be closed in Y. In this section singular sets are introduced whose definitions are motivated by the above and used to study some properties of mappings with respect to closedness.

DEFINITION 4.1. Let $f: X\to Y$ be a mapping and let

$$S_4=\{y\in Y: f^{-1}(y) \text{ is not compact}\}.$$

THEOREM 4.1. *A closed mapping $f: X\to Y$ is perfect if and only if $S_4=\phi$.*

PROOF. This follows from the definition.

Recall the following result on closed mappings.

LEMMA. *Let $f: X\to Y$ be a closed mapping and let U be a neighborhood of $f^{-1}(y)$, $y\in Y$. Then there exists a neighborhood W of y such that $f^{-1}(W)\subset U$.*

THEOREM 4.2. *If $f: X\to Y$ is a closed mapping with X locally compact, then S_4 is a closed set.*

PROOF. Suppose $\mathrm{cl}S_4-S_4\neq\phi$ and let $y\in\mathrm{cl}S_4-S_4$. There exists a neighborhood U of $f^{-1}(y)$ such that $\mathrm{cl}U$ is compact. Let W be a neighborhood of y such that $f^{-1}(W)\subset U$. Now, $W\cap S_4\neq\phi$ and for $y'\in W\cap S_4$, $f^{-1}(y')\subset f^{-1}(W)\subset\mathrm{cl}U$.

However, since $f^{-1}(y')$ is a closed subset of cl$U$, $f^{-1}(y')$ is compact. This contradiction proves cl$S_4-S_4=\phi$.

It is known that the restriction of a closed mapping to an inverse set is a closed mapping. Therefore, we get the following result.

THEOREM 4.3. *If $f:X\to Y$ is a closed mapping and $g=f|(X-f^{-1}(S_4))$, then g is a perfect mapping of $X-f^{-1}(S_4)$ onto $Y-S_4$.*

PROOF. Let $y\in Y-S_4$. Then $f^{-1}(y)\cap f^{-1}(S_4)=\phi$ and $f^{-1}(y)$ is compact in X. Hence, $g^{-1}(y)$ is a compact set in $X-f^{-1}(S_4)$. Hence, g is a closed mapping with compact point inverses.

The following result is due to Michael (2, Thm. 1.1).

THEOREM 4.4. *If $f:X\to Y$ is a closed mapping with X paracompact and Y locally compact or first countable, then the boundary of $f^{-1}(y)$,* Bdy$f^{-1}(y)$, *is compact for each $y\in Y$.*

THEOREM 4.5. *If $f:X\to Y$ is a closed non-perfect mapping with X paracompact and Y locally compact or first countable, then* Int$f^{-1}(y)\neq 0$ *for each $y\in S_4$.*

PROOF. If $y\in S_4$, then $f^{-1}(y)$ is not compact. Since Bdy$f^{-1}(y)$ is compact, $f^{-1}(y)-$Bdy$g^{-1}(y)\neq\phi$. Hence, Int$f^{-1}(y)\neq\phi$.

COROLLARY. *Under the conditions of Theorem 4.5, $f^{-1}(S_4)$ is not compact and* Int$f^{-1}(S_4)\neq\phi$.

THEOREM 4.6. *Let $f:X\to Y$ be an open-closed mapping with X paracompact and Y locally compact or first countable. Then S_4 is a discrete subspace of Y.*

PROOF. If $y\in S_4$, then Int$f^{-1}(y)\neq\phi$. Since f is an open mapping, $f($Int$f^{-1}(y))=\{y\}$ is an open set in Y. Thus Y Hausdorff implies $\{y\}$ is an open and closed subset in Y.

THEOREM 4.7. *Let $f:X\to Y$ be a closed non-perfect mapping with X paracompact and Y locally compact or first countable. Let $A=\{$*Int$f^{-1}(y):y\in S_4\}$.
Then $g=f|(X-A)$ is a perfect mapping onto $f(X-A)$.

PROOF. Since A is an open set, $X-A$ is closed and thus g is a closed mapping. For $y \in S_4$, either $g^{-1}(y)=\phi$ or $g^{-1}(y)=\text{Bdy} f^{-1}(y) \cap (X-A)$ which is compact. For $y \notin S_4$, $g^{-1}(y)=f^{-1}(y) \cap (X-A)$ which is also compact. Therefore, g is a perfect mapping.

THEOREM 4.8. *Let $f: X \to Y$ be a closed mapping such that $f^{-1}(y)$ is connected for each $y \in Y-S_4$ is a continuum if and only if $X-f^{-1}(S_4)$ is a continuum.*

PROOF. That $Y-S_4$ is compact and connected when $X-f^{-1}(S_4)$ is follows immediately from the continuity of the mapping f.

Assume $Y-S_4$ is a continuum. By Theorem 4.3, $f|(X-f^{-1}(S_4))$ is a perfect mapping onto $Y-S_4$. Thus, $X-f^{-1}(S_4)$ is compact. Now, suppose $X-f^{-1}(S_4)$ is not connected. Then there exist disjoint closed sets A and B such that $X-f^{-1}(S_4)=A \cup B$. Hence, $Y-S_4=f(A) \cup f(B)$ where $f(A)$ and $f(B)$ and closed sets. If $y \in f(A) \cap f(B)$, then

$$f^{-1}(y)=(f^{-1}(y) \cap A) \cup (f^{-1}(y) \cap B),$$

a separation of $f^{-1}(y)$. Since $f^{-1}(y)$ is connected it much be that $f(A) \cap f(B) =\phi$ and thus $f(A) \cup f(B)$ is a separation of $Y-S_4$. This contradiction proves $X-f^{-1}(S_4)$ is connected. Therefore, $X-f^{-1}(S_4)$ is a continuum.

It follows from the definition that $S_4 \subset S_3$. The following example shows that, in general, $S_4 \neq S_3$.

EXAMPLE 4.1. Let X be the space of real numbers with the usual topology and let $Y=[-1,1]$ be a subspace of X. Define $f: X \to Y$ such that

$$f(x)=x \text{ for } -1 \leq x \leq 1:$$

$$f(x)=\frac{1}{x} \text{ for } |x|>1.$$

Then f is a continuous finite-to-one mapping and $S=\phi$. Now let U be a neighborhood of $0 \in Y$. There exists $\varepsilon>0$ such that $\{y: -\varepsilon \leq y \leq \varepsilon\} \subset U$.

Since $\left\{x: x>\frac{1}{\varepsilon}\right\} \subset f^{-1}[-\varepsilon, \varepsilon]$,

$f^{-1}[-\varepsilon, \varepsilon]$ is not compact and hence $0 \in S_3$. Note that f is not a closed mapping since $f[1, \infty)=(0,1] \subset Y$.

THEOREM 4.9. *If $f: X \to Y$ is a closed mapping with X locally compact, then*

$S_4 = S_3$.

PROOF. For any $y \in Y - S_4$, let U be a neighborhood of $f^{-1}(y)$ such that cl$U$ is compact. Let W be any neighborhood of y such that $f^{-1}(W) \subset U$. Then for any compact set $K \subset W$, $f^{-1}(K) \subset \mathrm{cl}U$. Thus, $f^{-1}(K)$ is compact and $y \in Y - S_3$. Hence, $S_3 \subset S_4$ and this implies $S_4 = S_3$.

When a mapping fails to be closed, there exists a set of points in Y which can be described as a singular set with respect to closedness of the mapping.

DEFINITION 4.2. Let $f: X \to Y$ be a mapping. Let T be the set of all points $y \in Y$ such that each neighborhood of y contains a non-closed set with closed trace. T will be called the ***singular set of the mapping with respect to closedness.***

THEOREM 4.10. *Let $f: X \to Y$ be a mapping with Y regular. Then f is a closed mapping if and only if $T = \phi$.*

PROOF. If f is a closed mapping, then every set in Y with closed trace is closed. Thus, $T = \phi$.

Assume f is not a closed mapping. There exists a closed set $K \subset X$ such that $f(K)$ is not closed in Y. Let $y \in \mathrm{cl}f(K) - f(K)$. For any neighborhood U of y, there exists a neighborhood V of y such that $\mathrm{cl}V \subset U$ and $\mathrm{cl}V \cap f(K) \neq \phi$. The set $f^{-1}(\mathrm{cl}V) \cap K$ is closed and $f[f^{-1}(\mathrm{cl}V) \cap K] = [\mathrm{cl}V \cap f(K)]$. Since $y \in \mathrm{cl}[\mathrm{cl}V \cap f(K)] - [\mathrm{cl}V \cap f(K)]$, $\mathrm{cl}V \cap f(K)$ is not a closed set. Hence, $y \in T$.

THEOREM 4.11. *If $f: X \to Y$ is a mapping, then T is a closed set.*

PROOF. If f is closed, then $T = \phi$. If f is not closed, let $y \in \mathrm{cl}T$. For each neighborhood U of y, $U \cap T \neq \phi$. Let $y' \in U \cap T$. Then U is a neighborhood of y' and as such U contains a non-closed set with closed trace. Thus, $y \in T$.

In light of the proof of Theorem 4.10, another set which measures the non-closedness of the mapping f is introduced.

DEFINITION 4.3. Let $T_1 = \{y \in Y :$ there exists a closed set $K \subset X$ such that $y \in \mathrm{cl}f(K) - f(K)\}$.

THEOREM 4.12. *Let $f: X \to Y$ be a mapping. Then f is a closed mapping if and only if $T_1 = \phi$.*

PROOF. If f is a closed mapping, then for every closed set $K \subset X$, $\mathrm{cl} f(K) - f(K) = \phi$. Hence, $T = \phi$.

If f is not a closed mapping, there exists a closed set $K \subset X$ such that $f(K)$ is not closed. Thus, $\mathrm{cl} f(K) - f(K) \neq \phi$ and $T \neq \phi$.

THEOREM 4.13. *If $f: X \to Y$ is a mapping with Y regular, then $T_1 \subset T$.*

PROOF. Let $y \in T_1$ and let U be any neighborhood of y. Y regular implies there exists a neighborhood V of y such that $\mathrm{cl} V \subset U$. There exists a closed set $K \subset X$ such that $y \in \mathrm{cl} f(K) - f(K)$. Then $V \cap f(K) \neq \phi$.
Let $L = f^{-1}[\mathrm{cl}(f(K) \cap V)] \cap K$.
The set L is a closed set and $f(L) = f(K) \cap V \subset U$. Thus U contains a non-closed set with closed trace and thus $y \in T$.

Recall that Q_1 denotes a set of points in Y each of which is interior to a compact set that has a compact inverse under the mapping $f: X \to Y$ and $S_1 = Y - Q_1$.

THEOREM 4.14. *For a mapping $f: X \to Y$, $T \cap Q_1 = \phi$, i.e., $T \subset S_1$.*

PROOF. Suppose $y \in T \cap Q_1$. Let U be a neighborhood of y such that $\mathrm{cl} U$ is compact and $f^{-1}(\mathrm{cl} U)$ is compact. Since $y \in T$ there exists a non-closed set $K \subset U$ with a closed trace, say $L \subset X$. Now $L \cap f^{-1}(\mathrm{cl} U)$ is compact and

$$f(L \cap f^{-1}(\mathrm{cl} U)) = K.$$

Therefore, K is compact, but Y is Hausdorff and this implies K is closed. This contradiction proves $T \cap Q_1 = \phi$.

THEOREM 4.15. *If $f: X \to Y$ is a mapping such that $f^{-1}(y)$ is compact for each $y \in Y$ and Y is locally compact, then $T = S_1$.*

PROOF. It suffices to show that $S_1 \subset T$.

Let $y \in S_1$. There exists a neighborhood U of y such that $\mathrm{cl} U$ is compact and $f^{-1}(\mathrm{cl} U)$ is not compact. Now, the mapping $f | f^{-1}(\mathrm{cl} U)$ is not a closed mapping since otherwise $f^{-1}(\mathrm{cl} U)$ would be a compact set. Thus, there exists a closed set $K \subset f^{-1}(\mathrm{cl} U)$ with $f(K)$ not closed in $\mathrm{cl} U$. Therefore, K is closed in X and $f(K)$ is not closed in Y. Hence, $y \in T$ and $S_1 \subset T$.

THEOREM 4.16. *Let $f: X \to Y$ be a mapping with Y regular and let $C = f^{-1}(T)$.*

Then the mapping $g=f|(X-C)$ is a closed mapping.

PROOF. Suppose g is not a closed mapping. Then there exixsts a set K closed in $X-C$ such that $g(K)$ is not closed in $Y-T$. There is a closed set $L\subset X$ such that $K=L\cap(X-C)$.

Now, $\mathrm{cl}_{Y-T}g(K)-g(K)\neq\phi$.

Let $y\in\mathrm{cl}_{Y-T}g(K)-g(K)$.

Then $y\notin g(K)$ implies $y\notin f(L)$. Thus, in Y $y\in\mathrm{cl}_{Y-T}g(K)-f(L)$.

Since $\mathrm{cl}_{Y-T}g(K)\subset\mathrm{cl}_Y f(L)$, then $y\in\mathrm{cl}_Y f(L)-f(L)$.

This implies $y\in T_1\subset T$ contrary to $y\in Y-T$. Thus g is a closed mapping.

THEOREM 4.17. *Let $f: X\to Y$ be a finite-to-one open mapping where Yis a metric space. Then $X-C$ is metrizable.*

PROOF. $X-C$ is an open set so that g is an open mapping. By the previous theorem, $g: X-C\to Y-T$ is a closed mapping. Thus, g is an open-closed finite-to-one mapping onto a metric space. Since g is a perfect mapping, $X-C$ is paracompact and completely regular. With these conditions Arhangelskii (1) has shown that the inverse image of a metric space under an open-closed finite-to-one mapping is metrizable.

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