Kyungpook Math. J.
Volume 21, Number 1
June, 1981

A NEW CLASS OF COMPLEXES OF LINES IN ELLIPTIC SPACE S_3 WITH ONE OF THE PRINCIPAL SURFACES COINCIDENT WITH THE COORDINATE SUREACES

By M.A. Soliman and N.H. Abdel-All

1. Introduction

Consider the elliptic space S_3 with a moving frame in the form of a normalized polar-tetrahedron $T(\overline{A}_0, \overline{A}_1, \overline{A}_2, \overline{A}_3)$. The fundamental equations and the structural equations of that frame are

$$\left.\begin{array}{l} d\overline{A}_\alpha=\omega_\alpha^\beta \overline{A}_\beta, \text{ where } \omega_\alpha^\beta+\omega_\beta^\alpha=0 \\ D\omega^i=\omega^j \Lambda\omega_j^i,\ \ Dw_j^i=\omega_j^k\Lambda\omega_k^i-\omega^j\Lambda\omega^i \\ \qquad (\omega^i=\omega_0^i=-\omega_i^0,\ i,\ j,\ k=1,\ 2,\ 3) \end{array}\right\} \tag{1}$$

Here ω_β^α are Pffaf's forms, D denotes the operator of exterior differentiation and Λ is the exterior product, between forms.

The differential equations of the complex of lines generated by the ray $\overline{A}_0\overline{A}_3$ related to its canonical tetrahedron in the 2nd differential neighbourhood of its ray [3] are

$$\left.\begin{array}{r} \omega_3^1=k\omega^2, \\ dk=p\omega^2+\alpha\omega^1+\beta\omega_3^2, \\ \omega^3-k\omega_1^2=\alpha\omega^2+q\omega^1+\gamma\omega_3^2, \\ k\omega^3-\omega_1^2=\beta\omega^2+\gamma\omega^1+r\omega_3^2, \end{array}\right\} \tag{2}$$

where k is the invariant of the 1st differential neighbourhood and $p, \alpha, \beta, q, \gamma, r$ are the invariants of the 2nd differential neighbourhood of the ray of the complex.

2. Existance theorem

The differential equations of the complex of lines, whose one of the principal surfaces [1] coincident with the coordinate surface [3] are

$$\begin{array}{r} \omega_3^1=k\omega^2, \\ dk=p\omega^2, \end{array}$$

$$\left.\begin{aligned} \omega^3 - k\omega_1^2 &= q\omega^1 + \gamma\omega_3^2, \\ k\omega^3 - \omega_1^2 &= \gamma\omega^1 + r\omega_3^2. \end{aligned}\right\} \tag{3}$$

Exterior differentiation of $dk = p\omega^2$, gives

$$dp\Lambda\omega^2 + \frac{p(q-r)}{1-k^2}\,\omega^1\Lambda\omega_3^2 = 0,$$

since ω^1, ω^2 and ω_3^2 are linearly independent forms, we must have

$$p(q-r) = 0. \tag{4}$$

The cases for which $p=0$, $q-r\neq 0$ and $p\neq 0$, $q-r=0$ have been studied in [1] and [2] respectively.

In the previous paper [2], the equation $\omega^2=0$, determines a holonomic congruence has been proposed. In general this congruence is W-congruence (a congruence is called ***W-congruence*** if the asymptotic lines on the focal surfaces have the same equations) [3].

In the present paper we investigate the complex for which the congruence $\omega^2=0$, is not a W-congruence. For this purpose the asymptotic lines on the focal surfaces σ_0 and σ_3 described by the focal points $\overline{A}_0$ and $\overline{A}_3$ are determined by

$$d\overline{A}_0 \cdot d\overline{A}_2 = \omega^1\omega_2^1 + \omega^3\omega_2^3 = 0, \tag{5}$$

and

$$d\overline{A}_3 \cdot d\overline{A}_1 = \omega^1\omega^3 + \omega_1^2\omega_3^2 = 0, \tag{6}$$

respectively. From (5), (6) and (3), it follows that, the asymptotic lines on the focal surfaces $\sigma_\alpha(\alpha=0,3)$ have not the same equations if and only if at least one of the equalities

$$q - k\gamma = 0, \quad \gamma - kq = 0, \tag{7}$$

holds with the condition $q-r=0$, $p\neq 0$.

(i) We study here the complexes for which $q-r=0$, $p\neq 0$ and $q=k\gamma$. In this case the differential equations (3) take the form

$$\left.\begin{aligned} \omega_3^1 &= kw^2, \\ dk &= p\omega^2, \\ \omega^3 - kw_1^2 &= q(\omega^1 + kw_3^2), \\ kw^3 - \omega_1^2 &= q(kw^1 + \omega_3^2). \end{aligned}\right\} \tag{8}$$

Exterior differentiation of the last three equations of (8) and using Cartan's

Lemma [4], the system (8) becomes

$$\left.\begin{array}{l}\omega_3^1=kw^2,\ dk=pw^2,\\ \omega^3=\pm i\omega^1,\ \omega_1^2=\mp iw_3^2.\end{array}\quad (i=\sqrt{-1})\right\} \tag{9}$$

From equations (9) we have the followng existance theorem.

THEOREM **1.** *The complex* (9) *exists within one arbitrary function of one variable.*

3. Geometrical construction

The following Theorems give the geometrical construction of the complex (9).

THEOREM 2. *When the ray* $\overline{A}_0\overline{A}_3$ *generates the complex* (9), *the line* $\overline{A}_0\overline{A}_2$ *generates a plane tangent to the absolute.*

PROOF. From the differentials

$$\left.\begin{array}{r}d\overline{A}_0=\omega^1(\overline{A}_1\pm i\,\overline{A}_3)+\omega^2\overline{A}_2,\\ d\overline{A}_2=i\omega_3^2(\overline{A}_1\pm i\overline{A}_3)-\omega^2\overline{A}_0,\\ d(\overline{A}_1\pm i\overline{A}_3)=\pm ik\omega^2(\overline{A}_1\pm i\overline{A}_3).\end{array}\right\} \tag{10}$$

It follows that $\overline{A}_0\overline{A}_2$ generates the plane $\overline{\sigma}$ with tangential coordinates

$$\overline{\sigma}=(\overline{A}_0\overline{A}_2,\ \overline{A}_1\pm i\overline{A}_3), \tag{11}$$

it follows from (10) and (11) that,

$$d\overline{\sigma}\equiv 0,\ (\mathrm{mod}\,\overline{\sigma}),$$

this means that, the plane $\overline{\sigma}$ is stable and tangent to the absolute at the stable point $\overline{A}_1\pm i\overline{A}_3$.

From the differentials

$$\left.\begin{array}{l}d\overline{A}_1=-\omega^1\overline{A}_1+\omega_1^2\overline{A}_2+\omega_1^3\overline{A}_3,\\ d\overline{A}_3=-\omega^3\overline{A}_0+\omega_3^1\overline{A}_1+\omega_3^2\overline{A}_2,\end{array}\right\} \tag{12}$$

it follows that, the line $\overline{A}_1\overline{A}_3$ generates a normal congruence whose its rays cut orthogonally horosphere with centre at the point $\overline{A}_1\pm i\overline{A}_3$, and its focal points coincident with the stable points $\overline{A}_1\pm i\overline{A}_3$.

LEMMA 1. *The equation* $\omega^2=0$, *determines a holonomic congruence belongs to the complex* (9).

PROOF. Since $D\omega^2\equiv 0$, $(\mathrm{mod}\,\omega^2)$, this means that $\omega^2=0$, is integrable and determines a holonomic congruence. The focal points of this congruence are the centres of the ray $\overline{A}_0\overline{A}_3$ of the complex. From (10) and (12) it follows that, the focal point $\overline{A}_0$ describes a line $l=(\overline{A}_0, \overline{A}_1\pm i\overline{A}_3)$ and the focal point $\overline{A}_3$ describes horosphere with centre at the stable point $\overline{A}_1\pm i\overline{A}_3$.

If the point $\overline{A}_0$ is stable$(\overset{1}{\omega}=\overset{3}{\omega}=0)$ then:

$$d\overline{A}_3=\omega_3^2\overline{A}_2,\ d|\overline{A}_0\overline{A}_3|=\omega_3^2|\overline{A}_0\overline{A}_2|,$$

this means that, $\overline{A}_3$ describes a curve lying on the horosphere and the ray $\overline{A}_0\overline{A}_3$ generates a boundle of lines, i.e., the complex (9) is a two parametric family of the boundle of lines.

THEOREM 3. *We consider a horosphere whose centre lies on the absolute. We take unmoving line passing through the plane tangent to the absolute at the centre of the horosphere. Take this line and the horosphere as the focal surfaces for a congruence. One parametric family of these congruences construct the complex* (9).

PROOF. We choose a frame of reference in the form of a polar tetrahedron $T(\overline{A}_0, \overline{A}_1, \overline{A}_2, \overline{A}_3)$, such that the point $\overline{A}_1\pm i\overline{A}_3$ coincides the centre of the horosphere, the line $\overline{A}_1\overline{A}_3$ coincident with the normal to the horosphere and the points $\overline{A}_0$ and $\overline{A}_2$ place on the line polar conjugate with $\overline{A}_1\overline{A}_3$ which lies on the tangent plane to the absolute at the point $\overline{A}_1\pm i\overline{A}_3$. Moreover, let the unmoving line l and the horosphere described by the points $\overline{A}_0$ and $\overline{A}_3$ respectively be taken as the focal surfaces of a congruence. Then the differential equations of the congruence constructed above are

$$\omega_3^1=0,\ \overset{2}{\omega}=0. \tag{13}$$

by construction the line $\overline{A}_0\overline{A}_3$ coincident with the ray of the complex, it follows that, $\overset{1}{\omega}$, $\overset{2}{\omega}$, ω_3^1 and ω_3^2 are the principal forms for this complex and the forms ω_3^1 and $\overset{2}{\omega}$ must be connected by $\omega_3^1=k\overset{2}{\omega}$.

It's exterior differentiation gives (2). Since $\overline{A}_1\overline{A}_3$ generates a congruence of lines, then the forms $\overset{3}{\omega}$ and ω_1^2 must be depend only on the forms $\overset{1}{\omega}$ and ω_3^2. This leads to the equality

$$\alpha=\beta=0. \tag{14}$$

Also the centre of the horosphere is a stable, i.e.,

$$d\overline{A}_1\pm id\overline{A}_3\equiv 0,\ (\mathrm{mod}\,\overline{A}_1\pm i\overline{A}_3), \tag{15}$$

from (12) and (15) we get

$$q=r=\pm i, \ (i=\sqrt{-1}) \tag{16}$$

The relations (14) and (16) characterize the constructed complex (9).

(ii) The complex for which $q-r=0$, $p\neq 0$ and $\gamma=kq$. In this case the line $\overline{A}_1\overline{A}_3$ which is ploar conjugate to $\overline{A}_0\overline{A}_2$ generates a stable plane. The existance Theorem and geometrical construction can be given by similar way as the above.

(iii) The complex for which $q-r=0$, $p\neq 0$, $q=k\gamma$ and $\gamma=kq$. In this case the focal surfaces of the congruences generated by the lines $\overline{A}_0\overline{A}_2$ and $\overline{A}_1\overline{A}_3$ are degenerate to stable planes, as reveled from the foregoing results in (i) and (ii).

Dept. of Mathematics,
Faculty of Science,
Assiut Univ. Assiut, Egypt

REFERENCES

[1] Soliman, M.A. and Abdel-All, N.H., *Complexes of lines in elliptic space S_3 with one of the principal surfaces coincident with the coordinate surface*, Bulletin of the Calcutta Mathematical Society India (1979) (in press).

[2] Soliman, M. A. and Abdel-All, N. H., *Another class of complexes of lines in elliptic space S_3 with one of the principal surfaces coincident with the coordinate surface*, Bulletin of Faculty of Science, Assiut University, Assiut (under publication).

[3] Soliman, M.A., *Theory of complexes in S_3*, Ph.D. Thesis Kiev (1972).

[4] Kovansov, N.E., *Theory of complexes*, Kiev (1962).