Kyungpook Math. J.
Volume 21, Number 1
June, 1981.

MAXIMAL TOPOLOGIES INDUCING CONTINUOUS CLOSED MAPS

By Thomas Gearhart and Norman Levine

1. Introduction

Suppose $f: X \to (Y, \mathscr{U})$ is a surjection from a set X onto a topological space $(Y, \mathscr{U})$. Let T_{cc} denote the collection of topologies on X which make f continuous and closed. The collection T_{cc} is never void; in fact, the "weak" topology $\mathscr{T}_w = \{f^{-1}[U] : U \in \mathscr{U}\}$ is easily seen to be its smallest member. In this paper we address the question of existence of maximal or largest members of T_{cc}.

If $A \subseteq X$ we will use the notation $\mathscr{C}A$ to denote the complement of A.

2. Maximality in T_{cc}

LEMMA 2.1. *Suppose $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ are topological spaces and $f: X \to Y$ is a surjection. Then f is a closed map if and only if the following condition is satisfied: for each $y \in Y$ and each $O \in \mathscr{T}$, if $f^{-1}(y) \subseteq O$, then there exists $U \in \mathscr{U}$ with $y \in U$ such that $f^{-1}[U] \subseteq O$.*

This is well known and the proof is omitted.

THEOREM 2.2. *Suppose $f: X \to (Y, \mathscr{U})$ is a surjection and $\mathscr{T}_0 \in T_{cc}$. Let $\mathscr{T}_s = sup\{\mathscr{T} \in T_{cc} : \mathscr{T}_0 \subseteq \mathscr{T}\}$. If $f^{-1}(y)$ is compact in $(X, \mathscr{T}_s)$ for each $y \in Y$, then there exists a maximal member $\mathscr{T}^*$ of T_{cc} such that $\mathscr{T}_0 \subseteq \mathscr{T}^*$.*

PROOF. The collection $\mathfrak{A} = \{\mathscr{T} \in T_{cc}: \mathscr{T}_0 \subseteq \mathscr{T}\}$ is nonvoid and partially ordered by inclusion. Let $\mathscr{C} \subseteq \mathfrak{A}$, $\mathscr{C}$ a nonvoid chain. If we define $\mathscr{T}' = \sup\{\mathscr{T} : \mathscr{T} \in \mathscr{C}\}$, it is clear that $\mathscr{T}_0 \subseteq \mathscr{T}'$, $\mathscr{T}'$ makes f continuous, and $\mathscr{T} \subseteq \mathscr{T}'$ for each $\mathscr{T} \in \mathscr{C}$.

We now show that $\mathscr{T}'$ makes f a closed map. Suppose $f^{-1}(y)$ is contained in $O' \in \mathscr{T}'$. For each $x \in f^{-1}(y)$, there exist $O_1^x, O_2^x, \cdots, O_{n_x}^x$, belonging to topologies in $\mathscr{C}$, such that $x \in O_1^x \cap O_2^x \cap \cdots \cap O_{n_x}^x \subseteq O'$. Setting $O_x = O_1^x \cap O_2^x \cap \cdots \cap O_{n_x}^x$ and using the fact that $\mathscr{C}$ is linearly ordered, we conclude that $x \in O_x \subseteq O'$ where O_x belongs to a topology in $\mathscr{C}$.

AMS(MOS) subject classifications (19801). Primary 54C05, 54C10.

Now $f^{-1}(y) \subseteq \bigcup \{O_x : x \in f^{-1}(y)\} \subseteq O'$. Since $f^{-1}(y)$ is compact in $(X, \mathscr{T}_s)_r$, there exists $\{x_1, x_2, \cdots, x_n\} \subseteq f^{-1}(y)$ such that $f^{-1}(y) \subseteq O_{x_1} \cup O_{x_2} \cup \cdots \cup O_{x_n} \subseteq O'$. Setting $O = O_{x_1} \cup O_{x_2} \cup \cdots \cup O_{x_n}$ and again invoking the fact that $\mathscr{C}$ is linearly ordered, we have $f^{-1}(y) \subseteq O \subseteq O'$ where $O \in \mathscr{T}$ for some $\mathscr{T} \in \mathscr{C}$. But $\mathscr{T} \in T_{cc}$ and so there exists $U \in \mathscr{U}$ such that $y \in U$ and $f^{-1}[U] \subseteq O \subseteq O'$. So indeed $\mathscr{T}'$ makes f closed.

It now follows that $\mathscr{T}'$ is an upper bound for $\mathscr{C}$ in $\mathscr{A}$. Consequently, Zorn's Lemma assures the existence of a maximal member $\mathscr{T}^*$ of $\mathscr{A}$. This $\mathscr{T}^*$ must be maximal in T_{cc} as well, and the proof is complete.

COROLLARY 2.3. *Suppose* $f : X \rightarrow (Y, \mathscr{U})$ *is a surjection and* $\mathscr{T}_0 \in T_{cc}$. *If* $f^{-1}(y)$ *is finite for each* $y \in Y$, *then there exists a maximal member* $\mathscr{T}^*$ *of* T_{cc} *such that* $\mathscr{T}_0 \subseteq \mathscr{T}^*$.

The authors' attempts to remove the compactness hypothesis from Theorem 2.2 have been unsuccessful. However, if we drop the requirement that the maximal member of T_{cc} contain a given $\mathscr{T}_0$ in T_{cc}, then the compactness restriction on the point inverses can be deleted. The construction which yields this result is motivated by the following two observations:

(i) A naive attempt to construct a maximal member of T_{cc} would be to declare $\{E \subseteq X : f[E]$ is $\mathscr{U}$-closed$\}$ to be the family of "closed sets" for the desired topology. This fails because the family is not closed under intersections.

(ii) In an attempt to overcome this difficulty, we can try focusing on a subset A of X such that $f|_A : A \rightarrow Y$ is a bijection. In fact, for such an A, the collection $\{E \subseteq X : f[E \cap A] = f[E]$ and $f[E]$ is $\mathscr{U}$-closed$\}$ does serve as the family of closed sets for a member of T_{cc}. Unfortunately, this topology is not maximal.

A slight enlargement of the collection in (ii) leads to the successful construction.

DEFINITION 2.4. Suppose $f : X \rightarrow Y$ is a surjection. $A \subseteq X$ is called a *preimage selection* if and only if $f|_A : A \rightarrow Y$ is a bijection.

THEOREM 2.5. *Suppose* $f : X \rightarrow (Y, \mathscr{U})$ *is a surjection and* A *is a preimage selection. Then* $\mathscr{F} = \{E \subseteq X : f[E \cap A] \subseteq B \subseteq f[E]$ *implies* B *is* $\mathscr{U}$*-closed*$\}$ *is the family of closed sets for a topology* $\mathscr{T}(A)$ *on* X *which makes* f *continuous and closed.*

PROOF. $f[\phi \cap A] \subseteq B \subseteq f[\phi]$ implies $B = \phi$ and $f[X \cap A] \subseteq B \subseteq f[X]$ implies $B = Y$. Hence $\phi, X \in \mathscr{F}$.

Suppose $\{E_\alpha : \alpha \in \Delta\}$ is a subcollection of $\mathscr{F}$. We need to show that $\bigcap\{E_\alpha : \alpha \in \Delta\} \in \mathscr{F}$. To this end suppose $f[\bigcap\{E_\alpha : \alpha \in \Delta\} \cap A] \subseteq B \subseteq f[\bigcap\{E_\alpha : \alpha \in \Delta\}]$. If we can show that B is $\mathscr{U}$-closed, it will follow that $\bigcap\{E_\alpha : \alpha \in \Delta\} \in \mathscr{F}$.

Observe that for each $\alpha \in \Delta$, $f[E_\alpha \cap A] \subseteq B \cup f[E_\alpha \cap A] \subseteq f[E_\alpha]$. Since each $E_\alpha \in \mathscr{F}$, it follows that $B \cup f[E_\alpha \cap A]$ is $\mathscr{U}$-closed for each $\alpha \in \Delta$. Now, using the fact that f is injective on A, we have

$$\bigcap\{B \cup f[E_\alpha \cap A] : \alpha \in \Delta\} = B \cup [\bigcap\{f[E_\alpha \cap A] : \alpha \in \Delta\}]$$
$$= B \cup f[\bigcap\{E_\alpha \cap A : \alpha \in \Delta\}] = B \cup f[\bigcap\{E_\alpha : \alpha \in \Delta\} \cap A] = B.$$

As an intersection of $\mathscr{U}$-closed sets, B itself must be $\mathscr{U}$-closed.

Next take E_1, $E_2 \in \mathscr{F}$, and suppose $f[(E_1 \cup E_2) \cap A] \subseteq B \subseteq f[E_1 \cup E_2]$. Observe that for $i=1, 2$, $f[E_i \cap A] \subseteq B \cap f[E_i] \subseteq f[E_i]$. Since $E_i \in \mathscr{F}$ for $i=1, 2$, it follows that $B \cap f[E_i]$ is $\mathscr{U}$-closed for $i=1, 2$. But then since $(B \cap f[E_1]) \cup (B \cap f[E_2]) = B \cap (f[E_1] \cup f[E_2]) = B \cap f[E_1 \cup E_2] = B$, B is $\mathscr{U}$-closed. So E_1, $E_2 \in \mathscr{F}$ implies $E_1 \cup E_2 \in \mathscr{F}$.

Thus $\mathscr{F}$ is indeed the family of closed sets for a topology on X which we denote $\mathscr{T}(A)$.

Let F be $\mathscr{U}$-closed. $f[f^{-1}[F] \cap A] \subseteq B \subseteq f[f^{-1}[F]]$ implies $F \cap f[A] \subseteq B \subseteq F$; but since $f[A]=Y$, this means $B=F$ and we can conclude that $f^{-1}[F] \in \mathscr{F}$. Hence $\mathscr{T}(A)$ makes f continuous.

Also, from the very definition of $\mathscr{F}$, $E \in \mathscr{F}$ implies $f[E]$ is $\mathscr{U}$-closed. Hence $\mathscr{T}(A) \in T_{cc}$ and the proof is complete.

Observe that if S is any superset of the preimage selection A, then $f[S \cap A] \subseteq B \subseteq f[S]$ implies $B=Y$. Hence, S is closed in $(X, \mathscr{T}(A))$. In particular, A itself is closed, and the following terminology is motivated:

DEFINITION 2.6. Suppose $f : X \rightarrow (Y, \mathscr{U})$ is a surjection and A is a preimage selection. Then the topology $\mathscr{T}(A)$ of Theorem 2.5 is called the *closed-selection topology* determined by A.

THEOREM 2.7. *Suppose $f : X \rightarrow (Y, \mathscr{U})$ is a surjection and A is a preimage selection. Then the closed-selection topology $\mathscr{T}(A)$ is a maximal member of T_{cc}.*

PROOF. Suppose $\mathscr{T}(A) \subseteq \mathscr{T} \in T_{cc}$. We need only show that $\mathscr{T} \subseteq \mathscr{T}(A)$.

Take E to be $\mathscr{T}$-closed. Suppose $f[E \cap A] \subseteq B \subseteq f[E]$. If we can show that B is $\mathscr{U}$-closed, it will follow that E is $\mathscr{T}(A)$-closed.

$A\cup f^{-1}[B]$ is $\mathscr{T}(A)$-closed by the remark immediately preceding Definition 2.6, and hence it is $\mathscr{T}$-closed as well. But then it follows that $E\cap(A\cup f^{-1}[B])$ is $\mathscr{T}$-closed. Then, since $\mathscr{T}$ makes f a closed map, B is $\mathscr{U}$-closed because

$$f[E\cap(A\cup f^{-1}[B])]=f[(E\cap A)\cup(E\cap f^{-1}[B])]$$
$$=f[E\cap A]\cup f[E\cap f^{-1}[B]]=f[E\cap A]\cup(f[E]\cap B)$$
$$=f[E\cap A]\cup B=B.$$

The following example shows that not all maximal members of T_{cc} are closed-selection topologies.

EXAMPLE 2.8. Let $X=[0, 1]$, $Y=S^1$, $\mathscr{U}=$ the usual topology on S^1, and define $f: X\to Y$ by $f(t)=e^{2\pi it}$, $0\leq t\leq 1$. If $\mathscr{T}$ is the usual topology on X, then $\mathscr{T}\in T_{cc}$. By Corollary 2.3, there exists a maximal member $\mathscr{T}^*$ of T_{cc} such that $\mathscr{T}\subseteq\mathscr{T}^*$. We will show that $\mathscr{T}^*$ is not a closed-selection topology.

The only two preimage selections are $A_1=(0, 1]$ and $A_2=[0, 1)$. Now $\left[0, \frac{1}{2}\right]$ is $\mathscr{T}$-closed but not $\mathscr{T}(A_1)$-closed because $f\left[\left[0, \frac{1}{2}\right]\cap A_1\right]=f\left[\left(0, \frac{1}{2}\right]\right]$ is not $\mathscr{U}$-closed. Hence $\mathscr{T}$ is not contained in $\mathscr{T}(A_1)$ and we conclude $\mathscr{T}^*\neq\mathscr{T}(A_1)$. In a similar manner, we can show that the $\mathscr{T}$-closed set $\left[\frac{1}{2}, 1\right]$ is not $\mathscr{T}(A_2)$-closed, and $\mathscr{T}^*\neq\mathscr{T}(A_2)$.

3. Largest Members of T_{cc}

DEFINITION 3.1. Let $f: X\to Y$ be a surjection. The *kernel* of f is defined $K=\{x\in X : x'\neq x \text{ implies } f(x')\neq f(x)\}$. The *cokernel* of f is defined $K_c=\complement f[K]$.

THEOREM 3.2. *Suppose $f: X\to(Y, \mathscr{U})$ is a surjection and let $\mathscr{E}=\{E\subseteq X : f[E]$ is $\mathscr{U}$-closed$\}$. Then the following are equivalent:*

(i) Every subset of K_c is open in $(Y, \mathscr{U})$.

(ii) $\mathscr{E}$ is closed under arbitrary intersections.

(iii) T_{cc} has a largest member.

(iv) All closed-selection topologies are the same.

PROOF. (i) implies (ii). Suppose every subset of K_c is open in $(Y, \mathscr{U})$. Let $\{E_\alpha : \alpha\in\Delta\}$ be an arbitrary subcollection of $\mathscr{E}$. That $\bigcap\{E_\alpha : \alpha\in\Delta\}\in\mathscr{E}$ follows from the following observations:

(a) $f[K]\cup f[\bigcap\{E_\alpha : \alpha\in\Delta\}\cap\complement K]$ is $\mathscr{U}$-closed because its complement is a subset of K_c.

(b) $f[\bigcap\{E_\alpha : \alpha\in\Delta\}]=\bigcap\{f[E_\alpha] : \alpha\in\Delta\}\cap(f[K]\cup f[\bigcap\{E_\alpha : \alpha\in\Delta\}\cap\mathscr{C}K])$.

(ii) implies (iii). It is easy to see that ϕ, X, and finite unions of members of $\mathscr{E}$ are in $\mathscr{E}$. So the hypothesis that $\mathscr{E}$ is closed under arbitrary intersections assures that $\mathscr{E}$ serves as the family of closed sets for a topology on X. The simple verifications that this topology makes f continuous and closed and contains all other members of T_{cc} are left to the reader.

(iii) implies (iv). Suppose T^{cc} has a largest member, say $\mathscr{T}^*$. Then for any preimage selection A, $\mathscr{T}(A)\subseteq\mathscr{T}^*$. But $\mathscr{T}(A)$ is a maximal member of T_{cc} by Theorem 2.7, and hence we must conclude $\mathscr{T}(A)=\mathscr{T}^*$.

(iv) implies (i). Suppose that the topologies $\mathscr{T}(A)$ are identical as A ranges over all preimage selections. Take $U\subseteq K_c$. Let A be any preimage selection. Choose another preimage selection B so that $A\cap B\cap f^{-1}[U]=0$ and $A\cap f^{-1}[\mathscr{C}U]=B\cap f^{-1}[\mathscr{C}U]$. Then $A\cap B=A\cap B\cap(f^{-1}[U]\cup f^{-1}[\mathscr{C}U])=A\cap B\cap f^{-1}[\mathscr{C}U]=A\cap f^{-1}[\mathscr{C}U]$. Since A is $\mathscr{T}(A)$-closed, B is $\mathscr{T}(B)$-closed, and $\mathscr{T}(A)=\mathscr{T}(B)$, it follows that $A\cap B$ is $\mathscr{T}(A)$-closed. Then using the fact that $\mathscr{T}(A)$ makes f closed, $\mathscr{C}U$ must be $\mathscr{U}$-closed because $f[A\cap B]=f[A\cap f^{-1}[\mathscr{C}U]]=f[A]\cap\mathscr{C}U=Y\cap\mathscr{C}U=\mathscr{C}U$. Hence $U\in\mathscr{U}$.

The Ohio State University

REFERENCES

[1] Himmelberg, C. J., ***On the Product of Quotient Spaces***, American Mathematical Monthly 72, 1103–1106, 1965.

[2] Kelley, J. L., ***General Topology***, Van Nostrand, New York, 1955.

[3] Whyburn, G. T., ***Dynamic Topology***, American Mathematical Monthly 77, 556–570, 1970.