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A REMARK ON NEW CRITERIA FOR UNIVALENT FUNCTIONS

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0. Abstract

St. Ruscheweyh suggested new criteria for univalent functions and two problems. In this paper, we shall give the relation between new criteria and fractional calculus and some results for Ruscheweyh's problems in a sense.

1. Introduction

Let A denote the family of functions $f(z)$ analytic in the unit disk $U=\{|z|<1\}$ and normalized $f(0)=0$ and $f'(0)=1$. And let K_n denote the class of functions $f(z)\in A$ satisfying the following conditions

$$(1)\quad \mathrm{Re}\left[\frac{\{z^n f(z)\}^{(n+1)}}{\{z^{n-1}f(z)\}^{(n)}}\right] > \frac{n+1}{2} \quad (z\in U),$$

where $n\in N\cup\{0\}$. In particular, for $n=0$ the conditions (1) become

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \qquad (z\in U).$$

Therefore, the class K_0 equals the class $S^*(1/2)$ that denote the class of starlike functions of order 1/2.

Let $f*g(z)$ denote the Hadamard product of two functions $f(z)$, $g(z)\in A$, that is,

$$f*g(z)=\frac{1}{2\pi i}\int_{|\xi|=\rho<1} f(\xi)g(z/\xi)\frac{1}{\xi}\,d\xi,$$

and

$$(2)\qquad D^{\alpha}f(z)=\left\{\frac{z}{(1-z)^{\alpha+1}}\right\}*f(z) \qquad (\alpha\geqq -1).$$

Then, the relation (2) implies

$$(3)\qquad D^n f(z)=\frac{z\{z^{n-1}f(z)\}^{(n)}}{n!},$$

where $n\in N\cup\{0\}$.

With this notation (3) we have that the necessary and sufficient condition for a function $f(z)\in A$ to be in the class $K_0\equiv S^*(1/2)$ is

$$\operatorname{Re}\left\{\frac{D^1 f(z)}{D^0 f(z)}\right\} > \frac{1}{2} \qquad (z \in U),$$

the necessary and sufficient condition for a function $f(z) \in A$ to be in the class $K_1 \equiv K$ is

$$\operatorname{Re}\left\{\frac{D^2 f(z)}{D^1 f(z)}\right\} > \frac{1}{2} \qquad (z \in U),$$

and the necessary and sufficient condition for a function $f(z) \in A$ to be in the class K_n is

$$\operatorname{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > \frac{1}{2} \qquad (z \in U). \tag{4}$$

Moreover, in the notation (4) also a class K_{-1} can be defined as the family of functions $f(z) \in A$ satisfying the condition

$$\operatorname{Re}\left\{\frac{f(z)}{z}\right\} > \frac{1}{2} \qquad (z \in U).$$

2. The definition $D^{\alpha} f(z)$

In [1], S. Owa defined the fractional integral and derivative of order α as follows.

DEFINITION 1. The fractional integral of order α is defined by

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\xi)\, d\xi}{(z-\xi)^{1-\alpha}},$$

where $\alpha > 0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{\alpha-1}$ is removed by requiring $\ln(z-\xi)$ to be real when $(z-\xi)$ is greater than 0. Moreover,

$$f(z) = \lim_{\alpha \to 0} D_z^{-\alpha} f(z).$$

DEFINITION 2. The fractional derivative of order α is defined by

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\xi)\, d\xi}{(z-\xi)^{\alpha}},$$

where $0 < \alpha < 1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\alpha}$ is removed by requiring $\ln(z-\xi)$ to be real when $(z-\xi)$ is greater than 0. Moreover,

$$f(z) = \lim_{\alpha \to 0} D_z^{\alpha} f(z)$$

and

$$f'(z)=\lim_{\alpha\to 1} D_z^{\alpha} f(z).$$

THEOREM 1. *Let the function*

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n .$$

Then, for $0<\alpha<1$ *we have*

$$D^{\alpha}f(z)=\frac{1}{\Gamma(1+\alpha)} D_z^{\alpha}\{z^{\alpha-1}f(z)\},$$

$$D^{0}f(z)=\lim_{\alpha\to 0} D^{\alpha}f(z),$$

and

$$D^{1}f(z)=\lim_{\alpha\to 1} D^{\alpha}f(z).$$

PROOF. For $0<\alpha<1$, we have from (2)

$$D^{\alpha}f(z)=\left\{\frac{z}{(1-z)^{1+\alpha}}\right\}*\left(z+\sum_{n=2}^{\infty}a_n z^n\right)$$

$$=z+\sum_{n=2}^{\infty}\frac{(n-1+\alpha)(n-2+\alpha)\cdots(1+\alpha)}{(n-1)!}a_n z^n.$$

On the other hand, by means of Definition 2

$$\frac{z}{\Gamma(1+\alpha)} D_z^{\alpha}\{z^{\alpha-1} f(z)\}=\frac{z}{\Gamma(1+\alpha)} D_z^{\alpha}\left(z^{\alpha}+\sum_{n=2}^{\infty}a_n z^{n-1+\alpha}\right)$$

$$=z+\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha)}{(n-1)!\Gamma(1+\alpha)}a_n z^n$$

$$=z+\sum_{n=2}^{\infty}\frac{(n-1+\alpha)(n-2+\alpha)\cdots(1+\alpha)}{(n-1)!}a_n z^n.$$

Therefore, the theorem is established.

THEOREM 2. *Let the function*

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n.$$

Then, for $0<\alpha<1$ *we have*

$$D^{-\alpha}f(z)=\frac{z}{\Gamma(1-\alpha)} D_z^{-\alpha}\{z^{-\alpha-1} f(z)\},$$

$$D^{0} f(z)=\lim_{\alpha\to 0} D^{-\alpha} f(z),$$

and

$$D^{-1} f(z)=\lim_{\alpha\to 1} D^{-\alpha} f(z).$$

The proof of the theorem is given in much the same way as Theorem 1.

3. The classes K_α and $K_{-\alpha}$

Let $\tilde{A}$ denote the family of functions

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

analytic in the unit disk U. And let K_α and $K_{-\alpha}$ denote the classes of functions $f(z)\in\tilde{A}$ satisfying the following conditions

$$\mathrm{Re}\left[\frac{D_z^{\alpha+1}\{z^\alpha f(z)\}}{D_z^\alpha\{z^{\alpha-1}f(z)\}}\right] > \frac{1+\alpha}{2} \qquad (z\in U)$$

and

$$\mathrm{Re}\left[\frac{D_z^{1-\alpha}\{z^{-\alpha}f(z)\}}{D_z^{-\alpha}\{z^{-\alpha-1}f(z)\}}\right] > \frac{1-\alpha}{2} \qquad (z\in U)$$

for $0<\alpha<1$, respectively.

Hence, we have Theorem 3 and Theorem 4 from Theorem 1 and Theorem 2, respectively.

THEOREM 3. *The necessary and sufficient condition for a function $f(z)\in\tilde{A}$ to be in the class K_α, $0<\alpha<1$, is*

$$\mathrm{Re}\left\{\frac{D^{1+\alpha}f(z)}{D^\alpha f(z)}\right\} > \frac{1}{2} \qquad (z\in U).$$

THEOREM 4. *The necessary and sufficient condition for a function $f(z)\in\tilde{A}$ to be in the class $K_{-\alpha}$, $0<\alpha<1$, is*

$$\mathrm{Re}\left\{\frac{D^{1-\alpha}f(z)}{D^{-\alpha}f(z)}\right\} > \frac{1}{2} \qquad (z\in U).$$

THEOREM 5. *Let the function $f(z)$ belong to the family $\tilde{A}$ and satisfy the condition*

$$\sum_{n=2}^{\infty} n(n+2)|a_n|<1.$$

Then, for $0<\alpha<1$, the function $f(z)$ is in the class K_α.

PROOF. The hypothesis of the theorem

$$\sum_{n=2}^{\infty} n(n+2)|a_n|<1$$

implies the inequality

$$\frac{1-\sum_{n=2}^{\infty}\frac{\Gamma(n+2)}{2(n-1)!}|a_n|}{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{(n-1)!}|a_n|}>\frac{1}{2}.$$

Accordingly,

$$\begin{aligned}\operatorname{Re}\left\{\frac{D^{1+\alpha}f(z)}{D^{\alpha}f(z)}\right\}&=\operatorname{Re}\left\{\frac{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha+1)}{(n-1)!\Gamma(\alpha+2)}a_n z^{n-1}}{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha)}{(n-1)!\,\Gamma(\alpha+1)}a_n z^{n-1}}\right\}\\&\geqq\frac{1-\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha+1)}{(n-1)!\,\Gamma(\alpha+2)}|a_n|}{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha)}{(n-1)!\,\Gamma(\alpha+1)}|a_n|}\\&>\frac{1-\sum_{n=2}^{\infty}\frac{\Gamma(n+2)}{2(n-1)!}|a_n|}{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{(n-1)!}|a_n|}\\&>\frac{1}{2}.\end{aligned}$$

This proves that the function $f(z)$ is in the class K_{α} by means of Theorem 3.

The next result is given in much the same way as Theorem 5.

THEOREM 6. *Let the function $f(z)$ belong to the family $\tilde{A}$ and satisfy the condition*

$$\sum_{n=2}^{\infty}(2n+1)|a_n|<1.$$

Then, for $0<\alpha<1$, the function $f(z)$ is in the class $K_{-\alpha}$.

4. The Ruscheweyh's problems for the classes K_{α} and $K_{-\alpha}$

St. Ruscheweyh gave the following problems in [2].

PROBLEM 1. What can be said about the classes K_{α}, if we replace the natural number n in (4) by an arbitrary real number $\alpha\geqq 1$. Is it perhaps that $K_{\alpha}\subset K_{\beta}$ for $\alpha>\beta$?

PROBLEM 2. Is K_{α} closed under the Hadamard product?

The truth of Problem 2 is trivial for $\alpha=-1$ and was proved by St. Ruscheweyh and T. Sheil-Small in [3] for $\alpha=0,\ 1$.

Now, we give some results for Problem 1 in a sense.

THEOREM 7. *Let the function $f(z)$ belong to the class $K_{\alpha+\delta}$ and satisfy the condition*

$$\sum_{n=2}^{\infty}\frac{(2n+3\delta+4)\,\Gamma(n+\delta+1)}{(n-1)!\,\Gamma(\delta+3)}\,|a_n|<1$$

for $0<\alpha<1$ and $0<\alpha+\delta<1$. Then the function $f(z)$ is in the class K_α.

PROOF. The hypothesis of the theorem

$$\sum_{n=2}^{\infty}\frac{(2n+3\delta+4)\,\Gamma(n+\delta+1)}{(n-1)!\,\Gamma(\delta+3)}\,|a_n|<1$$

implies the inequality

$$\frac{1-\sum_{n=2}^{\infty}\frac{\Gamma(n+\delta+2)}{(n-1)!\,\Gamma(\delta+3)}|a_n|}{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+\delta+1)}{(n-1)!\,\Gamma(\delta+2)}|a_n|}>\frac{1}{2}.$$

Accordingly,

$$\begin{aligned}
\operatorname{Re}\left\{\frac{D^{1+\alpha}f(z)}{D^{\alpha}f(z)}\right\} &= \operatorname{Re}\left\{\frac{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha+1)}{(n-1)!\,\Gamma(\alpha+2)}a_n z^{n-1}}{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha)}{(n-1)!\,\Gamma(\alpha+1)}a_n z^{n-1}}\right\}\\
&\geqq \frac{1-\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha+1)}{(n-1)!\,\Gamma(\alpha+2)}|a_n|}{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha)}{(n-1)!\,\Gamma(\alpha+1)}|a_n|}\\
&> \frac{1-\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha+\delta+1)}{(n-1)!\,\Gamma(\alpha+\delta+2)}|a_n|}{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha+\delta)}{(n-1)!\,\Gamma(\alpha+\delta+1)}|a_n|}\\
&> \frac{1-\sum_{n=2}^{\infty}\frac{\Gamma(n+\delta+2)}{(n-1)!\,\Gamma(\delta+3)}|a_n|}{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+\delta+1)}{(n-1)!\,\Gamma(\delta+2)}|a_n|}\\
&> \frac{1}{2}.
\end{aligned}$$

This proves that the function $f(z)$ is in the class K_α with the aid of Theorem 3.

COROLLARY 1. *There exists the function $f(z)$ of the class $K_{\alpha+\delta}$ such that is* *in the class* K_α, *where* $0<\alpha<1$ *and* $0<\alpha+\delta<1$.

COROLLARY 2. *For the family of functions*

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

satisfying the following condition

$$\sum_{n=2}^{\infty} \frac{(2n+3\alpha-3\beta+4)\ \Gamma(n+\alpha-\beta+1)}{(n-1)!\ \Gamma(\alpha-\beta+3)} |a_n|<1,$$

if $0<\beta<\alpha<1$ *and* $0<2\alpha-\beta<1$, *then* $K_\alpha \subset K_\beta$.

THEOREM 8. *Let the function* $f(z)$ *belong to the class* $K_{-\alpha+\delta}$ *and satisfy the condition*

$$\sum_{n=2}^{\infty} \frac{(2n+3\delta+1)\ \Gamma(n+\delta)}{(n-1)!\ \Gamma(\delta+2)} |a_n|<1$$

for $0<\alpha<1$ *and* $0<\alpha+\delta<1$. *Then the function* $f(z)$ *is in the class* $K_{-\alpha}$.

COROLLARY 3. *There exists the function* $f(z)$ *of the class* $K_{-\alpha+\delta}$ *such that is in the class* $K_{-\alpha}$, *where* $0<\alpha<1$ *and* $0<\alpha+\delta<1$.

COROLLARY 4. *For the family of functions*

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

satisfying the following condition

$$\sum_{n=2}^{\infty} \frac{(2n+3\alpha-3\beta+1)\ \Gamma(n+\alpha-\beta)}{(n-1)!\ \Gamma(\alpha-\beta+2)} |a_n|<1,$$

if $0<2\alpha-\beta<1$, *then* $K_{-\alpha} \subset K_{-\beta}$.

The proofs of Theorem 8, Corollary 3, and Corollary 4 are given in much the same way as Theorem 7, Corollary 1, and Corollary 2, respectively.

Finally, we have the following results for Problem 2 in a sense.

THEOREM 9. *Let the function* $f(z)$ *belong to the family* $\tilde{A}$ *and satisfy the condition*

$$\sum_{n=2}^{\infty} n(n+2)|a_n|<1.$$

Then, for $0<\alpha<1$, *the Hadamard product* $f*f(z)$ *is in the class* K_α.

PROOF. The hypothesis of the theorem

$$\sum_{n=2}^{\infty} n(n+2)|a_n|<1$$

leads the inequalities $|a_n|<1$ and

$$\frac{1-\sum_{n=2}^{\infty}\frac{\Gamma(n+2)}{2(n-1)!}|a_n|}{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{(n-1)!}|a_n|}>\frac{1}{2}.$$

On the other hand, since the Hadamard product of $f(z)$ and $f(z)$ is given by

$$f*f(z)=z+\sum_{n=2}^{\infty} a_n^{\,2} z^n,$$

we have

$$\mathrm{Re}\left\{\frac{D^{1+\alpha}\{f*f(z)\}}{D^{\alpha}\{f*f(z)\}}\right\}=\mathrm{Re}\left\{\frac{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha+1)}{(n-1)!\,\Gamma(\alpha+2)}a_n^{\,2}z^{n-1}}{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha)}{(n-1)!\,\Gamma(\alpha+1)}a_n^{\,2}z^{n-1}}\right\}$$

$$\geqq\frac{1-\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha+1)}{(n-1)!\,\Gamma(\alpha+2)}|a_n|^2}{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha)}{(n-1)!\,\Gamma(\alpha+1)}|a_n|^2}$$

$$>\frac{1-\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha+1)}{(n-1)!\,\Gamma(\alpha+2)}|a_n|}{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+\alpha)}{(n-1)!\,\Gamma(\alpha+1)}|a_n|}$$

$$>\frac{1-\sum_{n=2}^{\infty}\frac{\Gamma(n+2)}{2(n-1)!}|a_n|}{1+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{(n-1)!}|a_n|}$$

$$>\frac{1}{2}.$$

Therefore, the Hadamard product $f*f(z)$ is in the class K_α with the aid of Theorem 3.

COROLLARY 5. *There exists the function $f(z)$ of the class K_α such that the Hadamard product $f*f(z)$ is in the class K_α, where $0<\alpha<1$.*

COROLLARY 6. *If the function $f(z)$ belongs to the class K_α and satisfies the condition*

$$\sum_{n=2}^{\infty} n(n+2)|a_n|<1,$$

then the Hadamard product $f*f(z)$ *is in the class* K_{α}, *where* $0<\alpha<1$.

THEOREM 10. *Let the function* $f(z)$ *belong to the family* $\tilde{A}$ *and satisfy the condition*

$$\sum_{n=2}^{\infty} (2n+1)|a_n|<1.$$

Then, for $0<\alpha<1$, *the Hadamard product* $f*f(z)$ *is in the class* $K_{-\alpha}$.

COROLLARY 7. *There exists the function* $f(z)$ *of the class* $K_{-\alpha}$ *such that the Hadamard product* $f*f(z)$ *is in the class* $K_{-\alpha}$, *where* $0<\alpha<1$.

COROLLARY 8. *If the function* $f(z)$ *belongs to the class* $K_{-\alpha}$ *and satisfies the condition*

$$\sum_{n=2}^{\infty} (2n+1)|a_n|<1,$$

then the Hadamard product $f*f(z)$ *is in the class* $K_{-\alpha}$, *where* $0<\alpha<1$.

The proofs of Theorem 10, Corollary 7, and Corollary 8 are given in much the same way as Theorem 9, Corollary 5, and Corollary 6, respectively.

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