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A STONE-WEIERSTRASS THEOREM FOR *GM*-SPACES WITH ORDER IDENTITY

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1. Introduction

It is proved in a paper by Edwards and Vincent-Smith [3] that if $A(X)$ is the space of all affine functions on X, where X is a compact subset of a locally convex space, and if L is a linear subspace of $A(X)$ containing the constant functions, and if L separates the extreme points of X and satisfies the "filtering condition" (for each $f\in A(X)$, if $g_1\leq f$ and $g_2\leq f$ with $g_1\in L$, and $g_2\in L$, then there exists $g\in L$ such that $g_1\leq g$, $g_2\leq g$ and $g\leq f$), then L is dense in $A(X)$.

We prove here that in the above, "filtering condition" can be replaced by a weaker condition: the property (I_1''). More specifically we prove that if E is a *GM*-space with order identity, and if L is a subspace of E containing the order identity, and if L separates the pure states of E and satisfies either Property (I_1'') or Property (I_1'), then L is dense in E.

Note that in $C(X)$, any sublattice satisfies both properties (I_1'') and (I_1'), and every subalgebra of $C(X)$ containing the constant functions satisfies the Property (I_1'), even though subalgebras containing the constant functions do not satisfy the "filtering condition".

In the following, we define some of the terms used in this paper. Those that are not defined here and need clarifications, should be referred to the text by Shaefer [7] and another paper by the author [6].

(1. 1) DEFINTION. Let E be an ordered locally convex space with a positive cone C that is always assumed to be closed.

(a) A subspace I of E is *perfect* if for each 0-neighborhood U in E and for every $a\in I$, there exist $b\in I$, $u\in U$ and $v\in U$ such that $-u-b\leq a\leq b+v$.

(b) A subspace I has *Property* (I_1') if for every positive element x of E and $a\in I$ with $a\leq x$ and for every 0-neighborhood U in E, there exist a positive element b in I and $u\in U$ such that $a\leq b\leq x+u$.

(c) A subspace I has *Property* (I_1'') if for every positive $x\in E$ and $a\in I$

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with $a \leq x$, and for every O-neighborhood U in E, there exist $b \in I$ and $u \in U$ such that $0 \leq b \leq x$ and $a \leq b+u$.

(d) A subspace I has *Property* (I_1) if for every $a \in I$ and positive element x of E with $a \leq x$, there exists a positive element b of E such that $a \leq b \leq x$.

(e) A subspace I has *Property* (I_2) if for all $a \in I$, positive elements x and y of E with $a \leq x+y$, there exist b and c in I such that $b \leq x$, $c \leq y$ and $a \leq b+c$.

(1.2) DEFINITION. Let E be an ordered vector space with positive cone C and let F be a convex subset of C. Then F is called a *face* of C if all the positive x and y are contained in F, whenever $\lambda x+(1-\lambda)y \in F$ for some $0<\lambda<1$.

(1.3) DEFINITION. An ordered normed space is *regularly ordered* if:

(a) for each x in E, and every positive number ε. there exists y in E with $-y \leq x \leq y$ and $\|y\| \leq \|x\|+\varepsilon$.

(b) for every x and y with $-x \leq y \leq x$, $\|y\| \leq \|x\|$.

A *GM-space* is a regularly ordered Banach space such that for any x and y in the open unit ball, there exists z in the open unit ball with $x \leq z$ and $y \leq z$.

(1.4) DEFINITION. Let E be an ordered normed space.

(a) An order unit e is called an *order identity* or *strong order unit* if the order interval $[-e, e]$ is the unit ball of E.

(b) A continuous positive linear form p of norm 1 is called a *pure state* if it generates an extreme ray in the dual positive cone.

2. Approximation theorem

(2.1) THEOREM. *Let E be a GM-space with order identity e and let L be a subspace of E such that $e \in L$. If p is a pure state of E, then $(p+L^0) \cap S \cap C'$ is a face of $S \cap C'$ if and only if* $\ker(p) \cap L$ *is perfect in L, where S is the unit sphere of E'.*

Proof. To prove the sufficiency, assume f and g are in $S \cap C'$, $0<\lambda<1$ and $\lambda f+(1-\lambda)g=p+h>0$ with $h \in L^0$. Then

$$0 \leq \lambda f|_L \leq p|_L.$$

Since $\ker(p) \cap L$ is a perfect subspace of L, we must have

$$\lambda f|_L=\mu p|_L$$

for some μ, and from $e \in L$, we have $\lambda=\mu$. Therefore, $\lambda f|_L=\lambda p|_L$ and hence, $f|_L=p|_L$, which shows $f \in (p+L^0) \cap C'$. Thus $(p+L^0) \cap S \cap C'$ is a face of $S \cap C'$.

For the necessity, note that it is equivalent to prove that the subspace generated by $p|_L$ is an order ideal in L'. Suppose $0 < g_1 \in L'$ with $g_1 \leq p|_L$. Then by the Bauer's theorem we can extend g_1 to a continuous positive linear form g. Hence we have $0 \leq g|_L \leq p|_L$, or equivalently, $0 \leq g \leq p+k$ for some $k \in L^0$.

Therefore, we have

$$p+k=\lambda((1/\lambda)g)+(1-\lambda)((1/(1-\lambda))(p+k-g)),$$

where $\lambda = g(e) = g_1(e) \neq 0$. It follows from the hypothesis that $(1/\lambda)g \in p+L^0$. Thus $g=\lambda p+h$ with $h \in L^0$ and $g_1=\lambda p|_L$, which is what was to be proved.

(2. 2) LEMMA. *Let E be a GM-space with order identity and let L be a subspace of E with Property (I_1'') or (I_1'), and $e \in L$. Then for any pure state p of E, $(p+L^0) \cap S \cap C'$ is a face of $S \cap C'$.*

Proof. Let $F=(p+L^0) \cap S \cap C'$. Then F is clearly convex. Let f and g be in $S \cap C'$ and $0<\lambda<1$. $\lambda f+(1-\lambda)g \in F$, then $\lambda f+(1-\lambda)g=p+h$ for some $h \in L^0$. Note that with any of the Properties (I_1') and (I_1''), L^0 has Property (I_2) (See (3. 2) in [6] and (5. 5) in [6]) and that $\lambda f+(1-\lambda)g-h$ is positive. Hence there exist h_1 and h_2 in L^0 such that $h_1+h_2=-h$, $0 \leq \lambda f+h_1$ and $0 \leq (1-\lambda)g+h_2$. It is clear then that we have $0 \leq \lambda f+h_1 \leq p$ which implies $\lambda f+h_1=\lambda p$.

Consequently, we have $f=p-(1/\lambda)h_1 \in (p+L^0) \cap S \cap C'$ and similarly for g.

(2. 3) LEMMA. *Let E be a GM-space with order identity e and let L be a subspace of E containing e. Then for any $f \in U^0 \cap C'$ and for fixed $c \in C$, we have*

$$\text{Sup } \{g(c) \mid g \in (f+L^0) \cap U^0 \cap C'\} = \text{Inf}\{f(x) \mid c \leq x,\ x \in L\}.$$

Proof. Let

$$\alpha = \text{Sup}\{g(c) \mid g \in (f+L^0) \cap U^0 \cap C'\}$$

and

$$\beta = \text{Inf}\{f(x) \mid c \leq x, x \in L\}.$$

Then we clearly have $\alpha \leq \beta$. Suppose $\alpha < \beta$ and let $2\varepsilon = \beta - \alpha$. We define a linear form on the subspace generated by L and c such that $g_0(x)=f(x)$ if $x \in L$ *and* $g_0(c)=\alpha+\varepsilon$. Note that if $c \in L$, there is nothing to prove. Now, define a sublinear function δ on E such that

$$\delta(z) = \text{Inf}\{f(x) \mid z \leq x,\ x \in L\}.$$

Since we have $g_0(z) \leq \delta(z)$ for all z in the subspace generated by L and c. we can extend g_0 to a linear form g on E such that $g(x) \leq \delta(x)$ for all $x \in E$

by the Hahn-Banach Theorem. Now if $d\in C$ then $g(-d)\leq\delta(-d)\leq 0$ and hence g is positive. Also note that g is continuous since it is positive. Thus $g\in(f+L^0)\cap U'^0\cap C'$. But $g(c)=\alpha+\varepsilon$ which is contrary to the definition of α.

(2. 4) LEMMA. *Let E be a GM-space with order identity e and let L be a subspace of E containing e. If the subspace L has Property* (I_1') *or Property* (I_1''), *then*

$$C\cap(L^0\cap(F-C'))^0\subseteq L^{00}=\bar{L},$$

where F is the linear hull of the set of all pure states.

Proof. Let $c\in(L^0\cap(F-C))^0$ be positive and define a function $\bar{c}$ on $U^0\cap C'$ such that

$$\bar{c}(f)=\operatorname{Inf}\{f(x)\mid c\leq x,\ x\in L\}$$
$$=\operatorname{Sup}\{g(c)\mid g\in(f+L^0)\cap U^0\cap C'\}.$$

Then it is clear that we have

$$\bar{c}(\lambda f)=\lambda\bar{c}(f),$$
$$\lambda\bar{c}(f)+(1-\lambda)\bar{c}(g)\leq\bar{c}(\lambda f+(1-\lambda)g),\quad 0<\lambda<1.$$

We claim that

$$\bar{c}(\lambda f+(1-\lambda)g)\leq\lambda\bar{c}(f)+(1-\lambda)\bar{c}(g).$$

Since, if $h\in(\lambda f+(1-\lambda)g+L^0)\cap U^0\cap C'$, then $h=\lambda f+(1-\lambda)g+k$ is positive and by applying Property (I_2) for L^0, we see that there exist k_1 and k_2 in L^0 such that $0\leq\lambda f+k_1$, $0\leq(1-\lambda)g+k_2$ and $k_1+k_2\leq k$. Note that

$$\lambda f+k_1\in(\lambda f+L^0)\cap U^0\cap C',$$
$$(1-\lambda)g+k_2\in((1-\lambda)g+L^0)\cap U^0\cap C'.$$

Since $L^0\cap C'=0$, we have

$$k=k_1+k_2 \text{ and } (\lambda f+k_1)(c)+(1-\lambda)(g+k_2)(c)=h(c).$$

Thus, we have $\bar{c}(\lambda f+(1-\lambda)g)\leq\lambda\ \bar{c}(f)+(1-\lambda)\bar{c}(g)$.

We are now ready to prove $c=\bar{c}$ on $U^0\cap C'$. Note first that $\bar{c}$ is the infimum of a set of continuous real valued functions on $U^0\cap C'$ and hence it is upper semicontinuous. Let

$$\alpha=\operatorname{Sup}\{\bar{c}(f)-f(c)\mid f\in U^0\cap C'\},$$
$$K=\{f\in U^0\cap C'\mid\bar{c}(f)-f(c)=\alpha\}.$$

Then K is a nonempty closed convex subset of $U^0\cap C'$ and we have $K\subseteq S\cap C'=\{f\mid f\geq 0,\ f(e)=1\}$. Since $\bar{c}$ is additive and positive homogeneous on $U^0\cap C'$, it is easy to check that K is an extreme subset of $S\cap C'$. Thus K contains a pure state q of E. Then $\alpha=\bar{c}(q)-q(c)=0$ since $c\in(L^0\cap(F-C'))^0$. Therefore $\bar{c}=c$ on $U^0\cap C'$, hence on C'. Now, suppose $h\in L^0$ with $h=h_1-h_2$ where h_1 and h_2 are positive. Then $\bar{c}(h_1)=\bar{c}(h_2)$ from the definition of c. Therefore $c\in L^{00}=\bar{L}$.

(2.5) THFOREM. *Let E be a GM-space with order identity e and let L be a subspace of E containing e. If the subspace L has either the Property* (I_1') *or* (I_1'') *and separates the pure states of E, then L is dense in E.*

Proof. In view of (2.4), it is enough to prove that $L^0 \cap (F-C') = \{0\}$. Suppose $h \in L^0 \cap (F-C')$, then $h=\sum_{i=1}^{n} \lambda_i p_i - f$ where $\lambda_i > 0$, $f \in C'$ and p_i's are pure states. Then we have

$$0 \leqslant \sum_{i=1}^{n} \lambda_i p_i - h.$$

Since if L has either of the Properties (I_1') or (I_1''), then by (3.2) in [6] and (5.5) in [6] L^0 has (I_2), we have for some $h_i \in L^0$, $0 \leq \lambda_i p_i + h_i$ and $\sum_{i=1}^{n} h_i = -h$. Now by (2.2), $(p_i + L^0) \cap S \cap C'$ is a face of $S \cap C'$.

Since L separates pure states of E, we must have $(p_i + L^0) \cap C' = \{p_i\}$. Thus $p_i + (1/\lambda_i) h_i$ is contained in $(p_i + L^0) \cap C' = \{p_i\}$ for every i. Therefore we must have $h=0$.

(2.6) COROLLARY. *Let X be a compact Hausdorff space and L be a sublattice of $C(X)$. If L contains the constant functions and separates points of X, then L is dense in $C(X)$.*

Proof. It is easy to show that any sublattice satisfies Property (I_1) and hence Property (I_1''). Therefore the above follows from (2.5).

(2.7). EXAMPLE. Let X be a compact Hausdorff space and let L be a subalgebra of $C(X)$. Then we know that the closure of L is a sublattice and hence has Property (I_1). It is easy to check that L has Property (I_1') if it contains constant functions.

(2.8) COROLLARY. (see [3]). *Let X be a compact convex subset of a locally convex space and let $A(X)$ be the space of all affine functions on X. Let L be a linear subspace of $A(X)$ containing the constant functions and suppose that for each $f \in A(X)$, the set*

$$H = \{g \in L \mid g > f\}$$

is a decreasing filtering family

$(g_1, g_2 \in H \Longrightarrow$ *there exists* $g \in H$ *such that* $g \leq g_1$ *and* $g \leq g_2)$.

If L separates the extreme points of X, then L is dense in $A(X)$.

Proof. Note that the above filtering conition implies the Property (I_1) and hence Property (I_1''). Hence the result follows from (2.5).

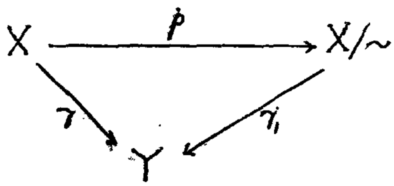
3. A representation of closed subspaces with Property (I_1'')

(3.1) PROPOSITION. *Let F and G be locally convex spaces and let X and Y be convex compact subsets of F and G respectively. If γ is a continuous affine function from X onto Y, then the range E of the induced map T from A(Y) into A(X) defined by $T(f)=f\circ\gamma$ is given by*

$$E=\{g\in A(X)\mid g(s)=g(t) \text{ for all } s\sim t\},$$

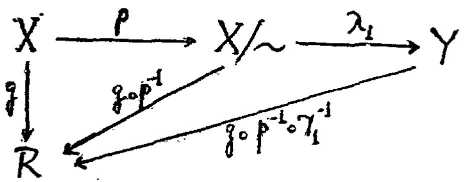
where $s\sim t$ if and only if $\gamma(s)=\gamma(t)$.

Proof. Note that we clearly have $T(A(Y))\subseteq E$. In order to prove that $E\subseteq T(A(Y))$, we consider the following decomposition of γ:



where p is the identification map and γ_1 is such that $\gamma_1\circ p=\gamma$. Since the set $\{(t,s)\mid t\sim s\}$ is closed in $X\times X$, we have $X/\sim$ is a compact Hausdorff space and γ_1 is a continuous bijection.

Now, suppose $g\in A(X)$ is such that $g(s)=g(t)$ for all $s\sim t$, then by considering the following diagram, we have $g\circ p^{-1}$ is continuous by the transgression Theorem (see Theorem VI. 6.3 in [2]).



Since γ_1 is a homeomorphism, the map $g\circ p^{-1}\circ\gamma_1^{-1}$ is also continuous. It is easy to check that $g\circ p^{-1}\circ\gamma_1^{-1}$ is affine and so $g\circ p^{-1}\circ\gamma_1^{-1}\in A(Y)$. Thus $g=g\circ p^{-1}\circ\gamma_1^{-1}\circ\gamma=T(g\circ p^{-1}\circ\gamma_1^{-1})$, and hance it is contained in the range of T.

(3.2) PROPOSITION. *Let E be a GM-space with order identity e. If a closed subspace L of E containing e has the Property (I_1'') then L is isometrically order isomorphic to A(Y) for some compact Hausdorff space Y.*

Proof. Let X be the set of all positive linear forms on E with norm 1. Then it is clear that E is isometrically order isomorphic to $A(X)$: the space of all continuous affine functions on X when X has the topology induced from $\sigma(E',E)$. We define an equivalence relation on X such that $s\sim t$ if and only if $f(s)=f(t)$ for all $f\in L$. Let $Y=X/\sim$. Note that Y is a compact Hausdorff topological space. If γ is the projection map from X onto $X/\sim$ and if T is the induced map from $A(Y)$ into $A(X)$ defined by $T(f)=f\circ\gamma$, then it is clear that T is an isometry and by (2.8), and L is contained in the range of $A(Y)$. It is also clear that T is an order isomorphism onto $T(A(Y))$.

We show next that $T^{-1}(L)$ has Property (I_1'') in $A(Y)$. Suppose g be a positive element of $A(Y)$ and $f\in L$ such that $T^{-1}(f)\leq g$. Then Tg is positive and $f\leq Tg$. By applying Property (I_1'') of L we conclude that there exist $h\in L$ such that $0\leq h\leq T(g)$ and $f\leq h+\varepsilon e_x$, where ε is any positive number and e_x is the constant function 1 on X. Therefore we have

$$0\leq T^{-1}(h)\leq g \text{ and } T^{-1}(f)\leq T^{-1}(h)+\varepsilon e_y.$$

Thus $T^{-1}(L)$ has Property (I_1'') in $A(Y)$ and it is closed in $A(Y)$ since L is closed. Note also that $T^{-1}(L)$ separates points of Y and hence by Theorem (2.5), we have $T^{-1}(L)=A(Y)$. Finally we have $L=T(A(Y))$, which completes the proof.

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