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ON SOME GRAPHS AND THEIR AUTOMORPHISM GROUPS

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1. Introduction

Let G be a finite group with identity e and let Ω be a set of generators for G with the properties:

$$\text{(i) } e \notin \Omega; \qquad \text{(ii) } g \in \Omega \Longrightarrow g^{-1} \in \Omega.$$

The *Caley graph* $\Gamma = \Gamma(G, \Omega)$ is the graph whose vertex-set and edge-set are

$$V\Gamma = G, \qquad E\Gamma = \{\{x, y\} \mid x^{-1}y \in \Omega\}.$$

It is well known that the Caley graph $\Gamma = \Gamma(G, \Omega)$ is vertex-transitive. In fact, the automorphism group AutΓ of Γ contains a regular subgroup $\bar{G}$ such that Aut $\Gamma = \bar{G}$ (Aut $\Gamma)_e$ and $\bar{G} \cong G$, where (Aut $\Gamma)_e$ is the stabilizer of the vertex e in Aut Γ.

For some Caley graphs the stabilizer (Aut$\Gamma)_e$ are isomorphic to a permutation group on Ω. In this paper we will study the automorphism groups Aut Γ of some Caley graphs Γ, where (Aut $\Gamma)_e$ are isomorphic to a permutation groups on Ω. We also discuss the t-tansitivity of these graphs.

The terminology of this paper is standard, and most of them are taken from [1]. A graph Γ is said to be *vertex-transitive* if its automorphsim group AutΓ acts transitively on the vertex-set $V\Gamma$. By a *t-arc* we mean an ordered set $[v_0, v_1, \cdots, v_t]$ of $t+1$ vertices such that each v_{i-1} is adjacent to v_i for $1 \leq i \leq t$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq t-1$. A graph Γ is said to be *t-transitive* $(t \geq 1)$ if Aut Γ acts transitively on the set of t-arcs in Γ.

2. Necessary lemmas

If a group G of automorphisms of a graph Γ acts transitively on the vertex-set $V\Gamma$, then we have $|G : G_x| = |V\Gamma|$, where G_x is the stabilizer of a vertex x in G. Moreover, the following holds.

LEMMA 2.1 *Let Γ be a vertex-transitive graph and let G be a group of automorphisms of Γ acting transitively on $V\Gamma$. For each vertex x, define*

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$$\Gamma_x = \{x \in V\Gamma \mid y \text{ is adjacent to } x\},$$
$$L_x = \{\sigma \in G_x \mid \sigma \text{ fixes all } y \text{ in } \Gamma_x\}.$$

Then L_x is normal in G_x, and G_x/L_x is isomorphic to a permutation group on Γ_x. In particular $|G_x:L_x| \leq k!$, where k is the valency of Γ.

If $y \in \Gamma_x$, then $|L_x : L_x \cap L_y| \leq (k-1)!$.

Proof. The stabilizer G_x acts on the set Γ_x. Hence there is a homomorphism of G_x into the symmetry group on Γ_x, with kernel L_x. Thus the first assertion holds. Note that $k = |\Gamma_x|$.

Since L_x and L_y are normal in $G_{xy} = G_x \cap G_y$, we have

$$L_x/(L_x \cap L_y) \cong L_xL_y/L_y \subseteq G_{xy}/L_y,$$

by the second isomorphism theorem. Clearly, G_{xy} acts on the set $\Gamma_y - \{x\}$. Therefore, there is a homomorphism of G_{xy} into the symmetric group of degree $k-1$. Hence the second assertion holds.

It is known that if Γ is t-transitive $(t \geq 2)$ then $L_x \cap L_y$ is a p-group for some prime p [3].

The following lemma is wellknown.

LEMMA 2.2 *Let Γ be a Caley graph $\Gamma(G, \Omega)$. Then the following hold.*

(1) *Γ is vertex-transitive and its valency is $|\Omega|$; in fact the automorphism group* Aut Γ *contains a regular subgroup $\bar{G}$ such that*

$$\text{Aut}\,\Gamma = \bar{G}(\text{Aut}\,\Gamma)_e \quad \textit{and} \quad \bar{G} \cong G,$$

where $(\text{Aut}\Gamma)_e$ *is the stabilizer of the vertex e (the identity of the group G).*

(2) *Let σ be an automorphism of the group G such that $\sigma(\Omega) = \Omega$. Then σ, regarded as a permutation of the vertex-set of Γ, is a group automorphism fixing e. Thus*

$$\Pi = \{\sigma \in \text{Aut}(G) \mid \sigma(\Omega) = \Omega\} \subseteq (\text{Aut}\Gamma)_e.$$

Proof. The proof can be found in [1, pp. 106-107].

Each $g \in G$ induces a permutation $\bar{g}$ of $V\Gamma = G$ given by $\bar{g}(x) = gx$. This permutation is an automorphism of Γ. Clearly, the subgroup $\bar{G} = \{\bar{g} \mid g \in G\}$ of AutΓ satisfies the conditions in (1). Note that $|\text{Aut}\Gamma: (\text{Aut}\Gamma)_e| = |G| = |\bar{G}|$ and $\bar{G} \cap (\text{Aut}\Gamma)_e = 1$. It is easy to prove (2).

By the definition of the Caley graph $\Gamma(G, \Omega)$, the set of all vertices adjacent to the vertex e is Ω. Hence the next lemma follows from Lemma 2.1.

LEMMA 2.3 *Let Γ be a Caley graph $\Gamma(G, \Omega)$. If the trivial automorphism is the only automorphism in* $(\text{Aut}\Gamma)_e$ *which pointwisely fixies the set Ω, then* $(\text{Aut}\,\Gamma)_e$ *is isomorphic to a permutation group on Ω.*

3. The automorphism groups of some Caley graphs

In this section we will explicitly determine the automorphism groups of some Caley graphs. Throughout this section, Lemma 2. 2 and Lemma 2. 3 will play important roles.

Let $G=\mathbf{Z}_m{}^n=\{(x_1,\cdots,x_n)\mid x_1,\cdots,x_n\in\mathbf{Z}_m\}$, where $m\geq 3$, $n\geq 2$ and $\mathbf{Z}_m$ is the additive group of integers modulo m. Let

$$\Omega=\{e_i,f_i\mid i=1,\cdots,n\},$$

where $f_i=-e_i$ and e_i has 1 in the ith coordinate and zeros elsewhere. Then the Caley graph $\Gamma=\Gamma(G,\Omega)$ is regular of valency $2n=|\Omega|$. The vertex-set of Γ is $G=\mathbf{Z}_m{}^n$ and two vertices are adjacent iff exactly one of their coodinates differ by ± 1 modulo m.

The following theorem gives the automorphism group $\mathrm{Aut}\Gamma$ of the Caley graph. By Lemma 2. 2, $\mathrm{Aut}\Gamma$ can be determined if the stabilizer $(\mathrm{Aut}\Gamma)_0$ of the vertex 0 is determined.

THEOREM 3. 1 *Let*

$$G=\mathbf{Z}_m{}^n,\quad m\geq 3,\quad n\geq 2,$$

and $\Omega=\{e_i,f_i\mid i=1,\cdots,n\}$ *as above. Let* Γ *denote the Caley graph* $\Gamma(G,\Omega)$. *Then* $\Pi=\{\sigma\in\mathrm{Aut}(G)\mid\sigma(\Omega)=\Omega\}$ *is isomorphic to the subgroup*

$$\mathcal{D}=\{BA\mid A \text{ is a permutation matrix and } B=\mathrm{diag}\{\varepsilon_1,\cdots,\varepsilon_n\} \text{ in } \mathrm{GL}_n(\mathbf{Z}_m),\ \text{where } \varepsilon_i=\pm 1\}$$

of $\mathrm{GL}_n(\mathbf{Z}_m)$.

Moreover, the following hold:

(1) *Let* $m\neq 4$. *Then* $(\mathrm{Aut}\Gamma)_0=\Pi$ *and*

$$\mathrm{Aut}\Gamma=\overline{G}\Pi=\{\overline{(y_1,\cdots,y_n)}\circ\sigma\mid(y_1,\cdots,y_n)\in G,\ \sigma\in\Pi\},\quad|\mathrm{Aut}\Gamma|=m^n2^nn!.$$

Each automorphism $\overline{(y_1,\cdots,y_n)}\circ\sigma$ *is given by*

$$(\overline{(y_1,\cdots,y_n)}\circ\sigma)\ (x_1,\cdots,x_n)=(x_1{}',\cdots,x_n{}'),\quad \text{where}$$

$$\begin{pmatrix}x_1{}'\\ \vdots\\ x_n{}'\end{pmatrix}=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}+[\sigma]\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix},\quad [\sigma]\in\mathcal{D}.$$

(2) *Let* $m=4$. *Then there exists an automorphism* $\pi\in(\mathrm{Aut}\Gamma)_0$ *of order* 3 *which fixes* 4^{n-1} *vertices*

$$(0,0,x_3,\cdots,x_n),\quad(0,-1,x_3,\cdots,x_n),$$
$$(2,1,x_3,\cdots,x_n),\quad(2,2,x_3,\cdots,x_n),\quad x_3,\cdots,x_n\in\mathbf{Z}_4,$$

and cyclically permutes 4^{n-1} 3-*subsets of* G

$$(1,0,x_3,\cdots,x_n),\quad(0,1,x_3,\cdots,x_n),\quad(-1,0,x_3,\cdots,x_n);$$
$$(1,-1,x_3,\cdots,x_n),\quad(0,2,x_3,\cdots,x_n),\quad(-1,-1,x_3,\cdots,x_n);$$
$$(-1,1,x_3,\cdots,x_n),\quad(2,0,x_3,\cdots,x_n),\quad(1,1,x_3,\cdots,x_n);$$
$$(-1,2,x_3,\cdots,x_n),\quad(2,-1,x_3,\cdots,x_n),\quad(1,2,x_3,\cdots,x_n);$$

where $x_3,\cdots,x_n$ *range over* $\mathbf{Z}_4$. *Moreover,*

$$(\mathrm{Aut}\Gamma)_0 = \langle \Pi, \pi \rangle$$

and it is isomorphic to the symmetric group $S(\Omega)$ *on* Ω. *We have*

$$\mathrm{Aut}\Gamma = \overline{G}(\mathrm{Aut}\Gamma)_0 = \overline{G}\langle \Pi, \pi \rangle, \quad |\mathrm{Aut}\Gamma| = 4^n \cdot (2n)!.$$

Proof. The proof is divided into the following four steps.

Step 1. *If* $\alpha \in (\mathrm{Aut}\Gamma)_0$ *fixes all vertices in* Ω, *then* α *is the trivial automorphism.*

Every vertex $x \in G = \mathbf{Z}_m{}^n$ can be uniquely expressed as $x = a_1e_1 + \cdots + a_ne_n$, where the a_i are the integers such that $0 \le a_i \le m-1$. Let $N(x) = a_1 + \cdots + a_n$. Then we have $0 \le N(x) \le n(m-1)$ and $\{x \in G \mid N(x) \le 1\} = \{0, e_1, e_2, \cdots, e_n\}$. We will prove Step 1 by induction on $N(x)$.

By the given condition α fixes all vertices x with $N(x) \le 1$ and it fixes $f_1, f_2, \cdots, f_n$. Let $i \ge 1$ and assume that α fixes all vertices y with $N(y) \le i$ and all vertices of the form $(i-1)e_j + f_k$, where $j, k \in \{1, 2, \cdots, n\}$ and $j \ne k$. Suppose that x is a vertex such that $N(x) = i+1$. Then the following two cases occur:

(a) There exist two distinct vertices y and z such that

$$x = y + e_j = z + e_k, \ \ e_j \ne e_k, \ \ N(y) = N(z) = i.$$

(b) $x = (i+1)e_j$ for some e_j.

Consider the case (a). By induction hypothesis α fixes both y and z, and so $\alpha(x)$ must be adjacent to both y and z. On the other hand, the only vertices adjacent to both y and z are x and $y - e_k = z - e_j$. Moreover, α fixes $y - e_k$ since $N(y - e_k) = i - 1$. Hence α fixes x.

Next consider the case (b). In this case $x = (i+1)e_j$ with $i \le m-2$. If $i = m-2$, then $x = f_j$ and so it is fixed by α. Now assume that $i < m-2$. Since α fixes the vertex ie_j, it induces a permutation on

$$\{ie_j + e_k, \ ie_j + f_k \mid k = 1, \cdots, n\},$$

which is the set of all vertices adjacent to ie_j. On the other hand, α fixes $ie_j + f_j$ and all vertices of the form $ie_j + e_k$ with $k \ne j$, by induction hypothesis and the case (a). Thus we have either $\alpha(x) = x$ or $\alpha(y) = x$ for some $y = ie_j + f_k$, where $k \ne j$. Note that y is adjacent to the vertex $z = (i-1)e_j + f_k$, which is fixed by α, by induction hypothesis. Therefore, if $\alpha(y) = x$ then x would be adjacent to z. But, this is a contradiction. Hence α fixes x.

The above arguments yield the fact that α fixes all verties $x \in G$. Thus α is the trivial automorohism.

Step 2. *The stabilizer* $(\mathrm{Aut}\Gamma)_0$ *is isomorphic to a subgroup of the symmetric group* $S(\Omega)$ *on* Ω. *We have* $\Pi \cong \mathcal{D}$ *and* $|\Pi| = 2^n(n!)$.

By Step 1 and Lemma 2.3 there is a monomorphism $\varphi\colon (\mathrm{Aut}\Gamma)_0 \to S(\Omega)$

given by $\varphi(\alpha)=\bar{\alpha}$, where $\bar{\alpha}=\alpha|_{\Omega}$ is the restriction of α on Ω.

By Lemma 2.2 we have $\Pi\subseteq(\mathrm{Aut}\Gamma)_0$. And the action of $\alpha\in\Pi$ on Ω is completely determined by the $\alpha(e_i)$ since $\alpha(f_i)=-\alpha(e_i)$. Hence we have $|\Pi|=(2n)(2n-2)\cdots 2=2^n(n!)$. On the other hand, the automorphism group $\mathrm{Aut}(G)$ of G is canonically isomorphic to the group $\mathrm{GL}_n(\mathbf{Z}_m)$. Now it is easy to show that $\Pi\cong\mathcal{D}$.

Step 3. *The assertion* (1) *holds.*

Let $m\neq 4$. By Step 2 we have $(\mathrm{Aut}\Gamma)_0\supseteq\Pi$ and $|(\mathrm{Aut}\Gamma)_0|\geq|\Pi|=2^n(n!)$. In order to prove (1), it suffices to show that $|(\mathrm{Aut}\Gamma)_0|=2^n(n!)$.

Since $(\mathrm{Aut}\,\Gamma)_0$ contains Π, it acts transitively on Ω. Hence it follows that $(\mathrm{Aut}\,\Gamma)_0=2n|(\mathrm{Aut}\Gamma)_{01}|$, where $(\mathrm{Aut}\Gamma)_{01}$ is the stabilizer of two vertices 0 and e_1. Clearly, $(\mathrm{Aut}\Gamma)_{01}$ acts on the set $\Omega_1=\Omega-\{e_1\}$ and it contains a subgroup $\Pi_1=\{\alpha\in\Pi\,|\,\alpha(e_1)=e_1\}$, which acts transitively on $\Omega-\{e_1,f_1\}$. Now suppose that $(\mathrm{Aut}\Gamma)_{01}$ acts transitively on Ω_1. Then there exists an automorphism $\pi\in(\mathrm{Aut}\,\Gamma)_0$ such that $\pi(e_1)=e_1$ and $\pi(e_2)=f_1$. This implies that $\pi(e_1+e_2)$ is adjacent to both e_1 and f_1. But, the only vertex adjacent to both e_1 and f_1 is 0 since $m\neq 4$. Therefore, there exists is no such π, and so $(\mathrm{Aut}\,\Gamma)_{01}$ does not act transitively on Ω_1. Thus we have

$$|(\mathrm{Aut}\Gamma)_0|=(2n)(2n-2)|(\mathrm{Aut}\Gamma)_{012}|,$$

where $(\mathrm{Aut}\Gamma)_{012}$ is the stabilizer of the vertices $0, e_1$ and e_2. The similar argument yields the fact that

$$|(\mathrm{Aut}\,\Gamma)_0|=(2n)(2n-2)\cdots 2\;|(\mathrm{Aut}\,\Gamma)_{01\cdots n}|,$$

where $(\mathrm{Aut}\Gamma)_{01\cdots n}$ is the stabilizer of $0, e_1, \dots, e_n$. Since $(\mathrm{Aut}\,\Gamma)_{01\cdots n}=\{1\}$ by Step 1, it follows that $|(\mathrm{Aut}\Gamma)_0|=2^n(n!)$.

Step 4. *The assertion* (2) *holds.*

Let $m=4$. Suppose that there is an automorphism π in $(\mathrm{Aut}\Gamma)_0$ such that

$$\pi(e_1)=e_2,\quad \pi(e_2)=f_1,\quad \pi(f_1)=e_1$$

and $\pi(e)=e$ for all e in the set $\Omega-\{e_1, e_2, f_1\}$. By an easy calculation we can show that π permutes vertices as shown in (2) and that this permutation is indeed an automorphism contained in $(\mathrm{Aut}\,\Gamma)_0$.

By Step 2, there is a monomorphism $\varphi:(\mathrm{Aut}\Gamma)_0\to S(\Omega)$ defined by $\varphi(\alpha)=\bar{\alpha}=\alpha|_{\Omega}$, Under this monomorphism, the subgroup Π is isomorphic to a subgroup

$$\bar{\Pi}=\langle(e_1\,e_i)(f_1 f_i),\ \ (e_i\,f_i)\,|\,i=1,\cdots,n\rangle$$

of $S(\Omega)$ and $\varphi(\pi)=\bar{\pi}=(e_1\,e_2\,f_1)$. Since

$$S(\Omega)=\langle(e_1\,e_i),\,(e_1 f_i)\,|\,i=1,\cdots,n\rangle,$$

the following equations imply that $\bar{\Pi}$ and $\bar{\pi}$ generate the group $S(\Omega)$, which is isomorphic to the symmetric group of degree $2n$.

$$\bar{\sigma}_2=\bar{\tau}\circ\bar{\pi}^{-1}\bar{\tau}^{-1}=(e_1\, e_2\, f_2),\ \text{ where } \bar{\tau}=(e_1\, e_2)(f_1 f_2);$$

$$\bar{\sigma}_i=\bar{\tau}_i\circ\bar{\sigma}_2\circ\bar{\tau}_i^{-1}=(e_1\, e_i\, f_i),\ \text{ where } \bar{\tau}_i=(e_2\, e_i)(f_2 f_i);$$

$$\bar{\sigma}_i\circ(e_i f_i)=(e_1\, e_i);\ \ (e_i f_i)\circ\bar{\sigma}_i=(e_1 f_i).$$

Hence $(\mathrm{Aut}\,\Gamma)_0=\langle\Pi,\pi\rangle\cong S(\Omega)$. Thus the assertion (2) holds.

COROLLARY 3.2 *The Caley graph* $\Gamma=\Gamma(G,\Omega)$ *defined in Theorem* 3.1 *is 2-transitive. But, it is not 3-transitive.*

Proof. Consider the 2-arc $[0,\ e_2,\ e_1+e_2]$. There are exactly $2n$-1 successors of this arc:

$$[e_2,\ e_1+e_2,\ e_1+e_2+e_i],\ \ 1\le i\le n;$$
$$[e_2,\ e_1+e_2,\ e_1+e_2+f_j],\ \ 2\le j\le n.$$

For each i, $1\le i\le n$, there exists an automorphism σ_i in Π such that

$$\sigma_i(e_2)=e_1,\ \ \sigma_i(e_1)=e_i,\ \ \sigma_i(e_i)=e_2.$$

Hence it follows that $\bar{e}_2\circ\sigma_i\in\bar{G}\Pi\subseteq\mathrm{Aut}\Gamma$ and

$$(\bar{e}_2\circ\sigma_i)[0,\ e_2,\ e_1+e_2]=[e_2,\ e_1+e_2,\ e_1+e_2+e_i].$$

Similarly, for each j, $2\le j\le n$, there exists $\bar{e}_2\circ\tau_j\in\bar{G}\Pi$ such that

$$(\bar{e}_2\circ\tau_j)[0,\ e_2,\ e_1+e_2]=[e_2,\ e_1+e_2,\ e_1+e_2+f_j].$$

Therefore, by a theorem of Tutte [5], the graph Γ is 2-transitive.

Suppose that Γ is 3-transitive. Then there exists an automorphism which sends a 3-arc $[0,\ e_1,\ e_1+e_2,\ e_2]$ into its successor $[e_1,\ e_1+e_2,\ e_2,\ 2e_2]$. This implies that there exists an automorphism $\sigma\in(\mathrm{Aut}\Gamma)_0$ such that

$$\sigma([0,\ e_1,\ e_1+e_2,\ e_2])=[0,\ e_2,\ f_1+e_2,\ f_1+2e_2].$$

But this contradicts to the fact that σ induces a permutation on the set Ω.

Therefore, Γ is not 3-transitive.

Obviously, Theorem 3.1 does not cover the case when $m=2$. If $m=2$, then $G=\mathbf{Z}_2{}^n$ and $\Omega=\{e_1,e_2,\cdots,e_n\}$ with $|\Omega|=n\ge 2$. In this case the Caley graph $\Gamma=\Gamma(G,\Omega)$ is called the n-cube and is written Q_n.

The automorphism group of Γ is well known [4]. In order to determine $\mathrm{Aut}\Gamma$, we may follow the steps of the proof of Theorem 3.1. In fact, the case (a) only occurs in Step 1. In Step 2, it is easy to see that $(\mathrm{Aut}\ \Gamma)_0$ is isomorphic to the symmetric group $S(\Omega)$ on Ω, since is a basis for the vector space $\mathbf{Z}_2{}^n$ over the field $\mathbf{Z}_2$. Thus we have

$$(\mathrm{Aut}\Gamma)_0\cong\mathscr{D}=\{A\in\mathrm{GL}_n(\mathbf{Z}_2)\,|\,A \text{ is a permutation matrix}\},$$
$$\mathrm{Aut}\Gamma=\bar{G}\,(\mathrm{Aut}\Gamma)_0=\{\overline{(y_1,\cdots,y_n)}\circ\sigma\,|\,(y_1,\cdots,y_n)\in G,\ \sigma\in(\mathrm{Aut}\Gamma)_0\}.$$

Each automorphism $\overline{(y_1,\cdots,y_n)}\circ\sigma$ is given by

$$((\overline{y_1,\cdots,y_n})\circ\sigma))\,(x_1,\cdots,x_n)=(x_1',\cdots,x_n'),\ \text{ where}$$

$$\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + [\sigma]\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad [\sigma] \in \mathcal{D}.$$

It is also well known that if $n \geq 3$ then the n-cube Q_n is 2-transitive, but not 3-transitive. If $n=2$, then Q_n is t-transitive for all $t \geq 1$.

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