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AZUMAYA ALGEBRAS OF A COMPLEX ANALYTIC SPACE

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We first recall the definition of an Azumaya algebra over a ringed space (X, O_X). We write O for the structure sheaf O_X, when there is no confusion. Undefined terms and the basic properties of Azumaya (central simple) algebras over a commutative ring can be found in [2] among many other sources. Let (X, O) be a ringed space and L be a sheaf of O-algebras. Then L is an Azumaya algebra over (X, O) if L is finitely presented as an O-module and L_x is an Azumaya O_x-algebra for every $x \in X$. (For the basic properties of sheaves of O-modules and O-algebras, see Chapter II. [5])

To define the Brauer group of a ringed space, we state the following three facts:

1. ([1], Theorem III. 4) *If M is a faithful locally projective sheaf of modules of finite type over a ringed space (X, O), then $\mathscr{H}\mathrm{om}_O(M, M)$ is an Azumaya algebra over (X, O).*
2. ([1], Theorem III. 5). *If A and B are Azumaya algebras over a ringed space (X, O), then so is $A \otimes B$.*
3. ([1], Theorem III. 6). *If A is an Azumaya algebra over a ringed space (X, O) and F is a sheaf of commutative O-algebras, then $A \otimes F$ is an Azumaya algebra over the ringed space (X, F).*

DEFINITION. ([6], 1.1.6). Let A and B be two Azumaya algebras over a locally ringed space (X, O). We say A and B are equivalent and write $A \sim B$ if there exist two faithful locally projective O-modules P and Q of finite type such that

$$A \otimes \mathscr{H}\mathrm{om}(P, P) \cong B \otimes \mathscr{H}\mathrm{om}(Q, Q).$$

Then $\sim$ is an equivalence relation.

Let $\mathrm{Br}(X)$ denote the set of equivalence classes of Azumaya algebras over (X, O). Write $[A]$ for the equivalence class of A in $\mathrm{Br}(X)$ and define a product in $\mathrm{Br}(X)$ by $[A][B] = [A \otimes B]$: this is well defined by Theorem II. 5 ([1]) and (2). It can be easily verified that $\mathrm{Br}(X)$ is an abelian group with

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$[0]=1$ and $[A]^{-1}=[A^{\circ}]$ because of (1).

Let (X, O_X) and (Y, O_Y) be ringed spaces. The ringed space (Z, O_Z) called disjoint union and denoted $X \amalg Y$, is defined as follows:

As a topological space Z is the disjoint union of X and Y, V is open set in Z if and only if $i^{-1}(V)$ and $j^{-1}(V)$ are open in X and Y respectively, where $i : X \to Z$ and $j : Y \to Z$ ane canonical inclusions. The structure sheaf O_Z is defined as follows; for any open set $V \subseteq Z$,

$$O_Z(V) := \{(s_1, s_2) \mid s_1 \in O_X(i^{-1}(V)),\ s_2 \in O_Y(j^{-1}(V))\}.$$

We often write X for a ringed space (X, O).

PROPOSITION 1. *Let X and Y be ringed spaces. Then*

$$\mathrm{Br}\ (X \amalg Y) \cong \mathrm{Br}(X) \oplus \mathrm{Br}(Y).$$

Proof. The maps

$$A \longmapsto (i^*(A),\ j^*(A))$$

$B \amalg C \longleftarrow\!\!\mid (B, C)$, where $B \amalg C$ is the sheaf obtained by glueing sheaves B and C,

give the equivalence between the category of sheaves of O_Z-algebras and the category of sheaves of $O_X \oplus O_Y$-algebras. From the definition of disjoint union of ringed spaces, and of Azumaya algebra, A is an Azumaya O_Z-algebra if and only if $i^*(A)$ and $j^*(B)$ are Azumaya algebras over X and Y respectively. Hence the conclusion follows.

Now we consider the following special kind of ringed space, namely complex analytic space ([4]). Let $\mathbf{C}$ be the field of complex numbers.

DEFINITION. ([10], p. 344). A ringed space (X, H) consisting of a topological space X and a sheaf H of algebras over $\mathbf{C}$ is called a complex analytic space if it satisfies the following condition; every $x \in X$ has a neighborhood U such that $(U, O_{X|U})$ is isomorphic, as a ringed space, to (W, O_{an}), where W is an open set in $\mathbf{C}^n$ (for the usual topology) and O_{an} is the sheaf of analytic functions on W.

For every $x \in X$, H_x is the local ring of germs of homomorphic functions at x, and (X, H) is a locally ringed space.

Let (X, O) be a scheme of finite type over $\mathbf{C}$ and let (X_h, H) be the associated complex analytic space (see [5], appendix B or [9]), i. e., X_h is the set of closed points of X whose topology is defined by taking the sets $V(U: f_1, \cdots, f_n : \varepsilon)$ as a basis of open sets, where $V(U: f_1, \cdots, f_n : \varepsilon)$ is defined as follows: for every open set U in X and finitely many regular functions $f_1, \cdots, f_n$ on U, and a number $\varepsilon > 0$,

$$V(U: f_1, \cdots, f_n : \varepsilon) = \{x \in X_h \cup U \mid |f_i(x)| < \varepsilon,\ \ i=1, 2, \cdots, n\}.$$

Then there is a continuous map

$$\varphi: X_h \longrightarrow X$$

of underlying topological spaces which maps X_h bijectively onto the set of closed points of X, and a natural morphism of structure sheaves

$$O \longrightarrow \varphi_*(H)$$

on X. In other words,

$$\varphi: (X_h, H) \longrightarrow (X, O)$$

is a morphism of locally ringed spaces.

Fix the morphism φ and consider the group homomorphism

$$\mathrm{Br}(X) \longrightarrow \mathrm{Br}(X_h)$$

induced by the contravariant functor $\mathrm{Br}(-)$ from the category of locally ringed spaces to the category of abelian groups.

We show that the map $\mathrm{Br}(X) \to \mathrm{Br}(X_h)$ is an isomorphism if X is a smooth projective variety. From now on we assume that X is a smooth projective variety.

THEOREM 2. *Let A and B be Azumaya algebras over a smooth projective variety (X, O) such that $\varphi^*(A)$ and $\varphi^*(B)$ are isomorphic. Then A and B are isomorphic as O-algebras.*

Proof. Let $f: \varphi^*(A) \longrightarrow \varphi^*(B)$ be a given isomorphism of H-algebras. Then there is a unique O-module isomorphism $g: A \to B$ such that $\varphi^*(g) = f$, by Theorem 2(§12 [9]). To show that g is in fact morphism of O-algebras, we consider the following commutative diagram of sheaves on X_h

$$\begin{array}{ccc} \varphi^{-1}(A) & \xrightarrow{\varphi^{-1}(g)} & \varphi^{-1}(B) \\ {\scriptstyle i}\downarrow & f & \downarrow{\scriptstyle j} \\ \varphi^*(A) & \longrightarrow & \varphi^*(B) \end{array}$$

where i and j are canonical $\varphi^{-1}(O)$-algebra homomorphism. Since $O_x \to H_x$ is injective for every $x \in X_h$, $\varphi^{-1}(O) \to H$ is injective. Since $\varphi^{-1}(A)$ and $\varphi^{-1}(B)$ are finitely generated locally free $\varphi^{-1}(O)$-module, i and j are injective. For every $x \in X_h$, the above diagram induces the following commutative diagram

$$\begin{array}{ccc} A_x & \xrightarrow{g_x} & b_x \\ {\scriptstyle i_x}\downarrow & f_x & \downarrow{\scriptstyle j_x} \\ A_x \otimes_{O_x} H_x & \longrightarrow & A_x \otimes_{O_x} H_x \end{array} .$$

Since i_x and j_x are injective and f_x is isomorphism as O_x-algebras, g_x is also O_x-algebra homomorphism. Hence the isomorphism g of sheaves of O-modules has to be O-algebra morphism, since g is morphism of sheaves and g_x is O_x-algebra morphism for every closed point in X.

From the above theorem, the following corollary follows.

COROLLARY 3. $\mathrm{Br}(X) \to \mathrm{Br}(X_h)$ *is one-to-one.*

To prove the equivalence from the set of Azumaya O-algebras to the set of Azumaya H-algebras, we consider the following functor F (functor F_2 in §2[11]).

Let P be a finitely generated locally free O-module. Then define

$$F\colon (\text{locally ringed spaces over a scheme } X) \longrightarrow (\text{Sets})$$

by

(*) $F((Y, O_Y)) := $ set of all multiplication laws m defined on $f^*(P)$ such that $(f^*(p), m)$ is an Azumaya algebra over (Y, O_Y),

where $f\colon (Y, O_Y) \to (X, O_X)$ is the structure morphism. Then F is a functor and a sheaf.

PROPOSITION 4. *Let functor*

$$G\colon (\text{locally ringed spaces over a scheme } X) \longrightarrow (\text{Sets})$$

be a sheaf. Suppose there exists an open cover $\{V_i\}_{i\in I}$ *of* X *such that the induced functor* G_i *is representable for all* $i\in I$. *Then* G *is representable.*

Proof. Let G_i be represented by Y_i. Then since G is a sheaf, Y_i can be glued together and hence G is represented by (Y, O_Y) obtained from Y_i's ([5] 3.5, Chapter 2).

From §2 in [11] and from the above proposition, the functor F defined by (*) is representable by a locally ringed space (Y, U) over X. Let f: $(Y, U) \to (X, O)$ be the structure map and $O \to f_*(U)$ be the morphism of structure sheaves on X. Then $f_*(U)_x$ is a smooth O_x-algebra, since U is locally smooth over O by construction ([11], §2) and since smoothness is preserved under localization ([7], 28.5).

THEOREM 5. *Let* A *be an Azumaya* H*-algebra. Then there exists an Azumaya* O*-algebra* B *such that* $\varphi^*(B) \cong A$.

Proof. By theorem 3([9], §12), there is a locally free O-module P of finite type such that $\varphi^*(P) \cong A$ as H-modules. Then from the above consideration and from the functor F defined by (*),

$$F(O) := \mathcal{H}om_{O\text{-alg}}(U, O)$$
$$F(H) := \mathcal{H}om_{O\text{-alg}}(U, \varphi_*(H))$$

for some locally smooth sheaf U of O-algebras. Now the theorem follows from the following proposition.

PROPOSITION 6. *Let U be a locally smooth O-algebra. Then the map*

$$\mathcal{H}om_{O\text{-alg}}(U, O) \to \mathcal{H}om_{O\text{-alg}}(U, \varphi_*(H))$$

is surjective.

Proof. By the adjoint property, the map

$$\mathcal{H}om_{H\text{-alg}}(\varphi^*(U),\ H) \to \mathcal{H}om_{O\text{-alg}}(U, \varphi_*(H))$$

is bijective. Again theorem 2([9], §12) says the map

$$\mathcal{H}om_{O\text{-mod}}(U, O) \longrightarrow \mathcal{H}om_{H\text{-mod}}(\varphi^*(U), H)$$

is bijective. Let $g:\varphi^*(U) \to H$ be an H-algebra homomorphism. Then consider the following commutative diagram as in the proof of theorem 2.

$$\begin{array}{ccc} \varphi^{-1}(U) & \xrightarrow{\ k\ } & \varphi^{-1}(O) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ \varphi_*(U) & \xrightarrow{\ g\ } & H \end{array},$$

k is the O-module homomorphism determined by g. Since U is locally smooth, U_x is flat O_x-module for every $x \in X_h$ and hence i_x and j_x are injective in the following induced commutative diagram.

$$\begin{array}{ccc} U_x & \xrightarrow{\ k_x\ } & O_x \\ {\scriptstyle i_x}\downarrow & & \downarrow{\scriptstyle j_x} \\ U_x \otimes H_x & \xrightarrow{\ g_x\ } & H_x \end{array}$$

As in the proof of theorem 2, k is O-algebra homomorphism and now the proposition follows.

From corollary 3 and theorem 5, we obtain the following.

COROLLARY 7. *If X is a projective variety and X_h is the associated analytic space, then the Brauer groups* $\mathrm{Br}(X)$ *and* $\mathrm{Br}(X_h)$ *are isomorphic.*

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