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ADDITIVE PERTURBATION OF M-ACCRETIVE OPERATORS IN THE SPACE $L^\infty(\Omega)$

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1. Introduction

Let Ω be a measure space with bounded measure, and let $\mathbf{R}$ be the set of all real numbers and $P(\mathbf{R})$ the power set of $\mathbf{R}$. Suppose that a nonlinear multivalued operator A is a m-accretive in the space $L^\infty(\Omega)$ and monotone in the space $L^2(\Omega)$, and that β is a multivalued operator from $\Omega\times\mathbf{R}$ into $P(\mathbf{R})$.

We define the operator $A+\beta$ from $L^\infty(\Omega)$ into $L^\infty(\Omega)$ by

$$A+\beta=\{[u,v+w]\in L^\infty(\Omega)\times L^\infty(\Omega):[u,v]\in A \text{ and } w(x)\in\beta(x,u(x)) \text{ a.e. } x\in\Omega\}.$$

We proved that $A+\beta$ is m-accretive when β is singlevalued ([5]). The objective of the present paper is to prove that $A+\beta$ is m-accretive in the space $L^\infty(\Omega)$ under certain hypotheses when β is multivalued.

§2 is devoted to some preliminary results concerning with the main result of this paper. §3 contains main results and a corollary.

2. Preliminaries

Let $L^p(\Omega)$ or only L^p denote the Lebesgue space of Ω furnished the norm $\|\cdot\|_p$ for $1\leqq p\leqq\infty$. Suppose that an operator A of $L^\infty(\Omega)$ satisfies the following condition (H1):

(H1) A is m-accretive in $L^\infty(\Omega)$ and monotone in $L^2(\Omega)$, that is to say,

(a) A is accretive in $L^\infty(\Omega)$: for all $[u_1,v_1]$, $[u_2,v_2]\in A$ and $\lambda>0$,
$$\|u_1-u_2+\lambda(v_1-v_2)\|_\infty\geqq\|u_1-u_2\|_\infty,$$

(b) A is monotone in $L^2(\Omega)$:
$$\text{for all } [u_1,v_1],[u_2,v_2]\in A,\ \int_\Omega(u_1-u_2)(v_1-v_2)\geqq 0,$$

(c) for all $\lambda>0$, $R(1+\lambda A)=L^\infty(\Omega)$.

We put $J_\lambda=(1+\lambda A)^{-1}$ and $A_\lambda=\lambda^{-1}(1-J_\lambda)$ for all $\lambda>0$. Then we have the following elementary properties.

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LEMMA 2.1. *Let A be an m-accretive operator of $L^\infty(\Omega)$. We have the following* ([3]).

(1) *A_λ is m-accretive in $L^\infty(\Omega)$ and for all $\lambda>0$ and $u_1, u_2 \in D(A)$,*
$$\|A_\lambda u_1 - A_\lambda u_2\|_\infty \leqq 2\lambda^{-1}\|u_1-u_2\|_\infty,$$
(2) *$A_\lambda \subset AJ_\lambda$ for all $\lambda>0$,*

(3) *$(A_\lambda)^{-1} = A^{-1}+\lambda I$ for all $\lambda>0$,*

(4) *$\|A_\lambda u\|_\infty \leqq \|Au\|_\infty$ for all $u\in D(A)$ and $\lambda>0$, where*
$$\|Au\|_\infty = \inf\{\|v\|_\infty : v\in Au\},$$
(5) *$J_\lambda u = J_\mu\left(\frac{\mu}{\lambda}u+\frac{\lambda-\mu}{\lambda}J_\lambda u\right)$ for all $u\in L^\infty(\Omega)$ and λ, $\mu>0$.*

LEMMA 2.2. *Suppose that an operator A of $L^\infty(\Omega)$ satisfies* (H1). *Then we have*

(1) *$A = A_2 \cap L^\infty(\Omega)\times L^\infty(\Omega)$, where A_2 is the closure of A in $L^2(\Omega)$.*

(2) *Let $[u_n, v_n]\in A$ such that $u_n \to u$, $v_n\to v$ in $\sigma(L^\infty(\Omega), L_1(\Omega))$ and $\overline{\lim}\int_\Omega u_n v_n \leqq \int_\Omega uv$ when $n\to+\infty$, then $[u,v]\in A$ and*
$$\int_\Omega u_n v_n \longrightarrow \int_\Omega uv \quad \text{when } n\to+\infty.$$

Suppose an operator β from $\Omega\times\mathbf{R}$ into $P(\mathbf{R})$ satisfies the following condition (H2):

(H2) $\begin{cases} \text{(a) a.e. } x\in\Omega,\ r\in\mathbf{R}\to\beta(x,r)\in P(\mathbf{R}) \text{ is maximal monotone,} \\ \text{(b) for every } r\in\mathbf{R}, \text{ there exists } u\in L^\infty(\Omega) \text{ such that } u(x)\in\beta(x,r) \\ \quad\ \text{a.e. } x\in\Omega. \end{cases}$

We define the operator β^{-1} from $\Omega\times\mathbf{R}$ into $P(\mathbf{R})$ by manner that $r\in\beta^{-1}(x,s)$ if and only if $s\in\beta(x,r)$. Finally, for all $\lambda>0$, let β_λ denote the operator from $\Omega\times\mathbf{R}$ into $\mathbf{R}$ by $\beta_\lambda=\lambda^{-1}(I-(I+\lambda\beta)^{-1})$, where I is the operator from $\Omega\times\mathbf{R}$ into $\mathbf{R}$ such that for all $x\in\Omega$, $I(x,r)=r$.

LEMMA 2.3. *Let β be an operator from $\Omega\times\mathbf{R}$ into $P(\mathbf{R})$. We have:*

(1) *if β satisfies* (H2)-(a), *then β^{-1} satisfies* (H2)-(a), *too,*

(2) *β satisfies* (H2)-(b) *if and only if for all $r\in\mathbf{R}$, there exists $u\in L^\infty(\Omega)$ such that $r\in\beta^{-1}(x,u(x))$* a.e. *$x\in\Omega$,*

(3) *if β satisfies* (H2)-(a), *then for all $\lambda>0$ and $r_1, r_2\in\mathbf{R}$,* a.e. *$x\in\Omega$, $r\in\mathbf{R}\to\beta_\lambda(x,r)\in\mathbf{R}$ is maximal monotone and*
$$|\beta_\lambda(x,r_1)-\beta_\lambda(x,r_2)|\leqq 2\lambda^{-1}|r_1-r_2|,$$
(4) *if β satisfies* (H2)-(b), *then for all $\lambda>0$, β_λ satisfies* (H2)-(b).

LEMMA 2.4. *Suppose that an operator β from $\Omega\times\mathbf{R}$ into $P(\mathbf{R})$ satisfies* (H2)-(a).

Then a.e. $x\in\Omega$, $\partial j(x,\cdot)=\beta(x,\cdot)$ *for some proper convex lowersemicontinuous function* $j:\Omega\times\mathbf{R}\to\,]-\infty,\infty]$ ([2]).

PROPOSITION 2.5. *Suppose that an operator* A *of* $L^\infty(\Omega)$ *satisfies* (H1) *and that an operator* β *from* $\Omega\times\mathbf{R}$ *into* $\mathbf{R}$ *satisfies* (H2) *and for every* $r_1, r_2\in\mathbf{R}$, *there exists* $M\geqq 0$ *such that*

$$|\beta(x,r_1)-\beta(x,r)|\leqq M|r_1-r_2| \text{ a.e. } x\in\Omega.$$

Let $A+\beta\neq\phi$. Then $A+\beta$ is m-accretive in $L^\infty(\Omega)$ ([5]).

3. M-accretivity of $A+\beta$ when β is multivalued

THEOREM 3.1. *Suppose that an operator* A *of* $L^\infty(\Omega)$ *satisfies* (H1) *and that a multivalued operator* β *from* $\Omega\times\mathbf{R}$ *into* $P(\mathbf{R})$ *satisfies* (H2), *and that for all* $f\in L^\infty(\Omega)$ *and* $\lambda>0$, *there exists at most an* $u\in L^\infty(\Omega)$ *such that* $u+\lambda(A+\beta)u\ni f$. *Let* $A+\beta\neq\phi$. *Then* $A+\beta$ *is m-accretive in* $L^\infty(\Omega)$.

Proof: By lemma 2.3, for all $\varepsilon>0$, $\beta_\varepsilon=\varepsilon^{-1}(I-(I+\varepsilon\beta)^{-1})$ satisfies (H2), and for all $r_1, r_2\in\mathbf{R}$, $|\beta_\varepsilon(x,r_1)-\beta_\varepsilon(x,r_2)|\leqq 2\,\varepsilon^{-1}|r_1-r_2|$.

Since $A+\beta$ is nonempty, $A+\beta_\varepsilon$ is nonempty, too. According to proposition 2.5, $A+\beta_\varepsilon$ is m-accretive in $L^\infty(\Omega)$ for all $\varepsilon>0$. Thus we have

(3.1) $\|(1+\lambda(A+\beta_\varepsilon))^{-1}f_1-(1+\lambda(A+\beta_\varepsilon))^{-1}f_2\|_\infty\leqq\|f_1-f_2\|_\infty$

for all $f_1, f_2\in L^\infty(\Omega)$ and $\lambda, \varepsilon>0$.

Let $\lambda>0$ and $f\in L^\infty(\Omega)$. Put $u_\varepsilon=(1+\lambda(A+B_\varepsilon))^{-1}f$ for all $\varepsilon>0$. Then $f\in u_\varepsilon+\lambda(A+\beta_\varepsilon)\,u_\varepsilon$ and $\|u_\varepsilon\|_\infty\leqq\|f\|_\infty$. Let

(3.2) $v_\varepsilon\in Au_\varepsilon$ such that $u_\varepsilon+\lambda(v_\varepsilon+\beta_\varepsilon u_\varepsilon)=f$.

Since $\beta_\varepsilon u_\varepsilon\in\beta(1+\varepsilon\beta)^{-1}$ and $\|(1+\varepsilon\beta)^{-1}u_\varepsilon\|_\infty\leqq\|u_\varepsilon\|_\infty\leqq\|f\|_\infty$ for all $\varepsilon>0$, by (H2)-(b), $\{\beta_\varepsilon u_\varepsilon\}_{\varepsilon>0}$ is bounded, and hence $\{u_\varepsilon\}_{\varepsilon>0}$ and $\{u_\varepsilon\}_{\varepsilon>0}$ are bounded in $L^\infty(\Omega)$ for each $\lambda>0$. Let $\{\varepsilon_n\}_{\varepsilon_n>0}$ such that

(3.3) $u_n=u_{\varepsilon_n}\to u$, $v_n=v_{\varepsilon_n}\to v$ in $\sigma(L^\infty(\Omega), L^1(\Omega))$ when $\varepsilon_n\to 0+$.

(3.4) $u_n=(1+\varepsilon_n\beta)^{-1}u_n=u_n-\varepsilon_n\beta_{\varepsilon_n}u\to u$ in $\sigma(L^\infty(\Omega), L^1(\Omega))$ when $\varepsilon_n\to 0+$.

By (H2)-(a) and lemma 2.4, there exists

$$j:\Omega\times\mathbf{R}\longrightarrow\,]-\infty,\infty]$$

such that j is proper convex lower-semicontinuous in $\mathbf{R}$ and a.e. $x\in\Omega$, $\partial j(x,\cdot)=\beta(x,\cdot)$.

We put, for all $u\in L^2(\Omega)$,

$$\phi(u)=\begin{cases}\int_\Omega j(u), & \text{if } j(u)\in L^1(\Omega)\\ +\infty, & \text{otherwise.}\end{cases}$$

Then ψ is a proper convex lower-semicontinuous function from $L^2(\Omega)$ into $]-\infty, \infty]$ and the subdifferential $\partial\psi$ of ψ coincide with the extension of β in $L^2(\Omega)$.

Since $\beta_{\varepsilon_n}u_n=\frac{1}{\lambda}(f-u_n)-v_n\in\beta(1+\varepsilon_n\beta)^{-1}u_n=\beta\bar{u}_n$, for all $z\in L^2(\Omega)$,

$$\begin{aligned}(3.5)\qquad \lambda\psi(z)-\lambda\psi(\bar{u}_n) &\geqq (f-(u_n+\lambda v_n), z-u_n+\varepsilon_n\beta_{\varepsilon_n}u_n)\\ &=(f, z-u_n+\varepsilon_n\beta_{\varepsilon_n}u_n)-(u_n+\lambda v_n, z)+(u_n+\lambda v_n, u_n+\varepsilon_n\beta_{\varepsilon_n}u_n).\end{aligned}$$

From (3.3), (3.4) and (3.5), we obtain

$$(3.6)\qquad \lambda\psi(z)-\lambda\psi(\bar{u}_n)\geqq (f, z-u)-(u+\lambda v, z)+\overline{\lim}(u_n+\lambda v_n, u_n)$$

when $n\to\infty$ for all $z\in L^2(\Omega)$. In particular $\psi(u)<\infty$, putting $z=u$ in (3.6) we have $\lim(u_n+\lambda v_n, u_n)\leqq(u+\lambda v, u)$.

Since A satisfies (H1), $1+\lambda A$ is m-accretive in $L^\infty(\Omega)$ and monotone in $L^2(\Omega)$. And $(u_n, u_n+\lambda v_n)\in 1+\lambda A$, $u_n+\lambda v_n\to u+\lambda v$, $u_n\to u$ in $\sigma(L^\infty(\Omega), L^1(\Omega))$ when $n\to+\infty$. Hence, by lemma 2.2,

$(u, u+\lambda v)\in 1+\lambda A$ and $\lim\limits_{n\to\infty}(u_n, u_n+\lambda v_n)=(u, u+\lambda v)$.

$(u_n-\varepsilon_n\beta_{\varepsilon_n}u, \frac{1}{\lambda}(f-u_n)-v_n)\in\beta$ and $u_n-\varepsilon_n\beta_{\varepsilon_n}u_n\to u$,

$\frac{1}{\lambda}(f-u_n)-v_n\to\frac{1}{\lambda}(f-u)-v$ in $\sigma(L^\infty(\Omega), L^1(\Omega))$ when $n\to+\infty$, and

$$\begin{aligned}&(u_n-\varepsilon_n\beta_{\varepsilon_n}u_n,\ \tfrac{1}{\lambda}(f-u)-v)\\ &\quad=\frac{1}{\lambda}(u_n-\varepsilon_n\beta_{\varepsilon_n}u, f)-\frac{1}{\lambda}(u_n, u_n+\lambda v_n)+\frac{1}{\lambda}(\varepsilon_n\beta_{\varepsilon_n}u_n, u_n+\lambda v_n)\\ &\quad\longrightarrow(u, f)-\frac{1}{\lambda}(u, u+\lambda v)\ \text{ when } n\to+\infty.\end{aligned}$$

Hence by the proposition 2.5 of [1], $(u, \frac{f-u}{\lambda}-v)\in\beta$ a.e. $x\in\Omega$. and hence u is a solution of $u+\lambda(A+\beta)u\ni f$. According to the hypothesis of uniqueness of this solution, we have proved that

$$f\in R(1+\lambda(A+\beta))\quad\text{and}\quad u_{\varepsilon_n}=(1+\lambda(A+\beta_{\varepsilon_n}))^{-1}f\to u=(1+\lambda(A+\beta))^{-1}f$$

in $\sigma(L^\infty(\Omega), L^1(\Omega))$ when $\varepsilon_n\to 0+$ $(n\to+\infty)$.

Let $f_1, f_2\in L^\infty(\Omega)$ and $\lambda>0$. Then according to (3.1), we have

$$\begin{aligned}&\|(1+\lambda(A+\beta))^{-1}f_1-(1+\lambda(A+\beta))^{-1}f_2\|_\infty\\ &\leqq\lim_{\varepsilon_n\to 0}\|(1+\lambda(A+\beta_{\varepsilon_n}))^{-1}f_1-(1+\lambda(A+\beta_{\varepsilon_n}))^{-1}f_2\|_\infty\leqq\|f_1-f_2\|_\infty.\end{aligned}$$

Consequently, we proved that $A+\beta$ is m-accretive in $L^\infty(\Omega)$ when β is multivalued.

DEFINITION 3.2. Let A be an operator of a space of measurable functions. We say that A is strongly injective if for all $[u_1, v_1]$, $[u_2, v_2]\in A$, $(u_1-u_2)(v_1-v_2)=0$ implies $u_1=u_2$.

LEMMA 3. 3. *Suppose that an operator A is monotone in $L^2(\Omega)$ and strongly injective, and that an operator β from $\Omega\times\mathbf{R}$ into $P(\mathbf{R})$ satisfies that* a. e. *$x\in\Omega$, $r\in\mathbf{R}\to\beta\ (x, r)\in P(\mathbf{R})$ is monotone. Then for all $f\in L^\infty(\Omega)$ and $\lambda>0$, there exists at most one $u\in L^\infty(\Omega)$ such that $u+\lambda(A+\beta)u\ni f$.*

Proof: Let $u_i\in L^\infty(\Omega)$ such that $u_i+\lambda(A+\beta)u_i\ni f\ (i=1,2)$.
By definition of $A+\beta$, there exists $v_i\in L^\infty(\Omega)$ such that $v_i\in Au_i$ and a. e. $x\in\Omega$, $\frac{1}{\lambda}(f-u_i)-v_i\in\beta u_i\ (i=1,2)$. Since A is monotone in $L^2(\Omega)$, we have

$$\int_\Omega (u_1-u_2)(v_1-v_2)\geqq 0,$$

and since β is monotone, we obtain

$$(u_1-u_2)\left(\frac{f-u_1}{\lambda}-v_1-\frac{1}{\lambda}(f-u_2)+v_2\right)\geqq 0,$$

and hence $(u_1-u_2)(v_1-v_2)\leqq 0$ a. e. $x\in\Omega$.
Hence $(u_1-u_2)(v_1-v_2)=0$. Since A is strongly injective, $u_1=u_2$.

From theorem 3. 1 and lemma 3. 3, we obtain the following corollary:

COROLLARY 3. 4. *Suppose that an operator A of $L^\infty(\Omega)$ satisfies* (H1), *and A is strongly injective in $L^\infty(\Omega)$, and that an multivalued operator β from $\Omega\times\mathbf{R}$ into $P(\mathbf{R})$ satisfies* (H2). *Let $A+\beta\neq\phi$.*
Then $A+\beta$ is m-accretive in $L^\infty(\Omega)$.

THEOREM 3. 5. *Suppose that an operator A of $L^\infty(\Omega)$ satisfies* (H1) *and that an operator β from $\Omega\times\mathbf{R}$ into $P(\mathbf{R})$ satisfies* (H2).
We suppose that for all $M\geqq 0$, the set $\{u\in D(A):\|u\|_\infty+\|Au\|_\infty\leqq M\}$ is relatively compact in $L^\infty(\Omega)$. Let $A+\beta\neq\phi$. Then there exists m-accretive operator C of $L^\infty(\Omega)$ such that $C\subset A+\beta$.

Proof: By proposition 2. 5, for all $\varepsilon>0$, $A+\beta_\varepsilon$ is m-accretive in $L^\infty(\Omega)$. Thus, for all $f\in L^\infty(\Omega)$, there exists $u_\varepsilon\in L^\infty(\Omega)$ such that

$$u_\varepsilon+\lambda(A+\beta_\varepsilon)u_\varepsilon=f.$$

Let $v_\varepsilon\in Au_\varepsilon$ such that $u_\varepsilon+\lambda(v_\varepsilon+\beta_\varepsilon u_\varepsilon)=f$. Put

$$J_{\lambda,\varepsilon}=(1+\lambda(A+\beta_\varepsilon))^{-1}.$$

Then we have $u_\varepsilon=J_{\lambda,\varepsilon}f$ and

$$(3.7)\qquad \|u_\varepsilon\|_\infty=\|J_{\lambda,\varepsilon}f\|_\infty\leqq\|f\|_\infty \text{ for all } \varepsilon>0.$$

Since $\{\beta_\varepsilon u_\varepsilon\}_{\varepsilon>0}$ is bounded, $\{u_\varepsilon\}_{\varepsilon>0}$, $\{v_\varepsilon\}_{\varepsilon>0}$ are bounded in $L^\infty(\Omega)$. Hence there exists $M\geqq 0$ such that $\|u_\varepsilon\|_\infty+\|Au_\varepsilon\|_\infty\leqq M$.
According to the hypothesis of relatively compact, $\{u_\varepsilon\}_{\varepsilon>0}$ is relatively com-

pact in $L^\infty(\Omega)$. Thus there exists $\{\varepsilon_i\}$ such that for all $f\in L^\infty(\Omega)$ and $\lambda>0$, $u_{\varepsilon_i}=J_{\lambda,\varepsilon_i}f \to J_\lambda f$ when $\varepsilon_i\to 0+$. By lemma 2.1 we have

$$J_{\lambda,\varepsilon_i}f=J_{\mu,\varepsilon_i}\left(\frac{\mu}{\lambda}f+\left(1-\frac{\mu}{\lambda}\right)J_{\lambda,\varepsilon_i}f\right) \tag{3.8}$$

for all $f\in L^\infty(\Omega)$ and $\lambda,\ \mu>0$.

Since $\|J_{\lambda,\varepsilon_i}f_1-J_{\lambda,\varepsilon_i}f_2\|_\infty\leqq\|f_1-f_2\|_\infty$ for all $f_1,f_2\in L^\infty(\Omega)$ and $\lambda>0$, $\varepsilon_i>0$, we obtain

$$J_\lambda f=J_\mu\left(\frac{\mu}{\lambda}f+\left(1-\frac{\mu}{\lambda}\right)J_\lambda f\right)$$

for all $f\in L^\infty(\Omega)$ and $\lambda,\mu>0$.

We define an operator C of $L^\infty(\Omega)$ by

$$C=\left\{\left[J_\lambda f,\frac{f-J_\lambda f}{\lambda}\right]: f\in L^\infty(\Omega)\ \ \textit{and}\ \ \lambda>0\right\}.$$

Then C is m-accretive in $L^\infty(\Omega)$ and $J_\lambda=(1+\lambda C)^{-1}$ for all $\lambda>0$.

For proof of $C\subset A+\beta$, let $[u,v]\in C$ and $u+v=f$. Then

$$u=(1+C)^{-1}f=\lim_{\varepsilon_i\to 0}(1+A+\beta_{\varepsilon_i})^{-1}f.$$

Put $u_{\varepsilon_i}=(1+A+\beta_{\varepsilon_i})f$. Then $f\in u_{\varepsilon_i}+Au_{\varepsilon_i}+\beta_{\varepsilon_i}u_{\varepsilon_i}$ and $\|u_{\varepsilon_i}\|_\infty\leqq\|f\|_\infty$. Let $v_{\varepsilon_i}\in Av_{\varepsilon_i}$ such that $u_{\varepsilon_i}+(v_{\varepsilon_i}+\beta_{\varepsilon_i}u_{\varepsilon_i})=f$. Since $\{\beta_{\varepsilon_i}u_{\varepsilon_i}\}_{\varepsilon_i>0}$ is bounded, $\{v_{\varepsilon_i}\}_{\varepsilon_i>0}$ is bounded in $L^\infty(\Omega)$, and $\{u_{\varepsilon_i}\}_{\varepsilon_i>0}$ is bounded, too. Hence there exists $\{\varepsilon_{i_n}\}_{\varepsilon_{in}>0}$ such that $u_{\varepsilon_{in}}\to u$ in $L^\infty(\Omega)$ and $v_{\varepsilon_{in}}\to v'$ in $\sigma(L^\infty(\Omega),\ L^1(\Omega))$ when $\varepsilon_{i_n}\to 0+$ and thus

$$\overline{\lim_{\varepsilon_{in}\to 0+}}\int_\Omega u_{\varepsilon_{in}}v_{\varepsilon_{in}}=\int_\Omega uv'.$$

According to lemma 2.2., we have $[u,v']\in A$. On the other hand,

$$\beta^{-1}{}_{\varepsilon_{in}}(f-u_{\varepsilon_{in}}-v_{\varepsilon_{in}})\ni u_{\varepsilon_{in}}.$$

Thus by lemma 2.1

$$\beta^{-1}(f-u_{\varepsilon_{in}}-v_{\varepsilon_{in}})+\varepsilon_{i_n}(f-u_{i\varepsilon_{in}}-v_{\varepsilon_{in}})\ni u_{\varepsilon_{in}}.$$

That is, $[u_{\varepsilon_{in}}-\varepsilon_{i_n}(f-u_{\varepsilon_{in}}-v_{\varepsilon_{in}}),f-u_{\varepsilon_{in}}v_{\varepsilon_{in}}]\in\beta$, and

$$f-u_{\varepsilon_{in}}-v_{\varepsilon_{in}}\to f-u-v' \text{ in } \sigma(L^\infty(\Omega),\ L^1(\Omega)),$$

$$u_{\varepsilon_{in}}-\varepsilon_{i_n}(f-u_{\varepsilon_{in}}-v_{\varepsilon_{in}})\to u \text{ in } L^\infty(\Omega)$$

when $\varepsilon_{i_n}\to 0+$, and thus

$$\overline{\lim_{\varepsilon_{in}\to 0+}}\int_\Omega(f-u_{\varepsilon_{in}}-v_{\varepsilon_{in}})(u_{\varepsilon_{in}}-\varepsilon_{i_n}(f-u_{\varepsilon_{in}}-v_{\varepsilon_{in}}))=\int_\Omega(f-u-v')u.$$

Thus by proposition 2.5 of [1], we obtain $f-u-v'\in\beta u$, and hence $v=f-u\in v'+\beta u\subset(A+\beta)u$, that is, $[u,v]\in A+\beta$.

The proof of theorem 3.5 is completed.

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