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COMMUTATIVITY OF THE FUNDAMENTAL GROUPS

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The concept of the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G) is a generalization of the fundamental group of a topological space. In this paper we define a H-structure of the transformation group (X, G) and investigate some properties of the fundamental group $\sigma(X, x_0, G)$ of (X, G).

Let (X, G) be a transformation group. Given any element g of G, a path f of order g with base point x_0 is a continuous map $f: I\rightarrow X$ such that $f(0)=x_0$ and $f(1)=gx_0$, where $I=[0, 1]$. Unless specially stated otherwise, it will be assumed that all paths have the same base point x_0. A path f_1 of order g_1 and a path f_2 of order g_2 give rise to a path $f_1+g_1f_2$ of order $g_1 g_2$ defined by the equation

$$(f_1+g_1f_2)(s)=\begin{cases} f_1(2s), & 0\leq s\leq \frac{1}{2}, \\ g_1f_2(2s-1), & \frac{1}{2}\leq s\leq 1. \end{cases}$$

Two paths f and f' of the same order g are said to be homotopic if there is a continuous map $F: I^2\rightarrow X$ such that

$$F(s, 0)=f(s), F(s, 1)=f'(s), 0\leq s\leq 1,$$
$$F(0, t)=x_0, \quad F(1, t)=gx_0, \quad 0\leq t\leq 1.$$

The notation $f\sim f'$ will be used to denote that f and f' are homotopic paths of the same order. The homotopic relation '$\sim$' is an equivalence relation on the set of all paths of the same order; hence the set of paths of order $g\in G$ is divided into homotopy classes. The homotopy class of a path f of order g will be denoted by $[f:g]$. Two homotopy classes of paths of different orders g_1 and g_2 are distinct, even if $g_1x_0=g_2x_0$.

If the two paths f_1 and f_1' of order g_1 are homotopic, and the two paths f_2 and f_2' of order g_2 are homotopic, then the two paths $f_1+g_1f_2$ and $f_1'+g_1f_2'$ are homotopic. Thus the equation

$$[f_1:g_1]*[f_2:g_2]=[f_1+g_1f_2 : g_1g_2]$$

gives a well-defined rule of composition for homotopy classes of paths of pre-

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scribed order. This composition $*$ is associative. Let e be the identity element of the group G. If x_0' denotes the constant map $x_0':I\to x_0$, where $x_0\in X$, then $[x_0':e]$ is an the identity element for this rule of composition. Furthermore, if ρ denotes the map from I to I which maps s to $1-s$, then given any homotopy class $[f:g]$ of order g, $[g^{-1}f\rho : g^{-1}]$ is a homotopy class of order g^{-1}, and

$$[f:g]*[g^{-1}f\rho : g^{-1}] = [g^{-1}f\rho : g^{-1}]*[f:g] = [x_0':e].$$

Thus the set of homotopy classes of paths of prescribed order with the rule of composition defined above is a group.

This group will be denoted by $\sigma(X, x_0, G)$ and will be called the fundamental group of a transformation group (X, G) with base point x_0 [1].

Let (X, G) be a transformation group and λ be a path in X from x_0 to x_1. We define a map $\lambda_*:\sigma(X,x_0,G)\to\sigma(X,x_1,G)$ by the equation $\lambda_*[f:g]=[\lambda\rho+f+g\lambda:g]$. Then the map λ_* is a group isomorphism of $\sigma(X,x_0,G)$ to $\sigma(X,x_1,G)$ and will be called the induced isomorphism by the path λ[1].

Let (X, G) and (Y, G) be transformation groups. The map $\phi:(X,G)\to(Y,G)$ is called an equivariant map if there is a continuous map $\phi:X\to Y$ such that $\phi(gx)=g\phi(x)$ for all $g\in G$ and $x\in X$.

1. DEFINITION. A transformation group (X, G) has H-structure if there is an continuous map $\psi:X\times X\to X$ such that

(1) $\psi(p,p)=p$ for some $p\in X$,

(2) $\psi(gx,y)=g\psi(x,y)=\psi(x,gy)$ for $g\in G$ and $x,y\in X$,

(3) The maps $f':X\to X$ defined by $f'(x)=\psi(x,p)$ and $f'':X\to X$ defined by $f''(x)=\psi(p,x)$ are both homotopic to the identity map on X by homotopies that leave gp fixed for $g\in G$

2. THEOREM. *Let (Y, G) be a transformation group with X path connected and G abelian, and (X, G) has H-structure. Then the transformation group (X, G) has an abelian fundamental group.*

Proof. Let $[f_1:g_1]$ and $[f_2:g_2]$ be two elements of $\sigma(X,p,G)$. Then we must show that $f_1+g_1f_2$ and $f_2+g_2f_1$ are homotopic. Let h_1' be the homotopy between f' and the identity and h_2' between f'' and the identity, and let $h_1=g_1h_1'$ and $h_2=g_1h_2'$ Then h_1 is a homotopy between g_1f' and $g_1\,id$, and h_2 is a homotopy between g_1f'' and $g_1\,id$. In fact,

$h_1(x,0)=g_1h_1'(x,0)=g_1\psi(x,p)$, $h_1(x,1)=g_1x$, for $x\in X$,

$h_2(x,0)=g_1h_2'(x,0)=g_1\psi(p,x)$, $h_2(x,1)=g_1x$, for $x\in X$,

$h_1(p,t)=g_1h_1'(p,t)=g_1p=g_1h_2'(p,t)=h_2(p,t)$ for all $t\in I$, and

$h_1(gp,t)=gh_1(p,t)$, $h_2(gp,t)=gh_2(p,t)$ for $g\in G$ and $t\in I$

Define a map $F:I^3\to X$ by the equations

$F(x, y, 0)=\psi(f_1(x), g_1f_2(y))$,
$F(x, 0, z)=h_1(f_1(x), z)$, $F(x, 1, z)=h_1(g_2f_1(x), z)$,
$F(0, y, z)=h_2(f_2(y), z)$, $F(1, y, z)=h_2(g_1f_2(y), z)$,

Then F is well-defined and continuous;

$$F(0,0,z)=\begin{cases} h_1(f_1(0), z)=h_1(p, z)=g_1p, \\ h_2(f_2(0), z)=h_2(p, z)=g_1p, \end{cases}$$
$$F(1,0,z)=\begin{cases} h_1(f_1(1), z)=h_1(g_1p, z)=g_1^2p, \\ h_2(g_1f_2(0), z)=h_2(g_1p, z)=g_1^2p, \end{cases}$$
$$F(1,1,z)=\begin{cases} h_1(g_2f_1(1), z)=h_1(g_2g_1p, z)=g_2g_1^2p, \\ h_2(g_1f_2(1), z)=h_2(g_1g_2p, z)=g_2g_1^2p, \end{cases}$$
$$F(0,1,z)=\begin{cases} h_1(g_2f_1(0), z)=h_1(g_2p, z)=g_2g_1p, \\ h_2(f_2(1), z)=h_2(g_2p, z)=g_2g_1p, \end{cases}$$
$$F(x,0,0)=\begin{cases} \psi(f_1(x), g_1f_2(0))=g_1\psi(f_1(x), p), \\ h_1(f_1(x), 0)=g_1\psi(f_1(x), p), \end{cases}$$
$$F(1,y,0)=\begin{cases} \psi(f_1(1), g_1f_2(y))=g_1^2\psi(p, f_2(y)), \\ h_2(g_1f_1(y), 0)=g_1^2\psi(p, f_2(y)), \end{cases}$$
$$F(x,1,0)=\begin{cases} \psi(f_1(x), g_1f_2(1))=g_1g_2\psi(f_1(x), p), \\ h_1(g_2f_1(x), 0)=g_1g_2\psi(f_1(x), p), \end{cases}$$
$$F(0,y,0)=\begin{cases} \psi(f_1(0), g_1f_2(y))=g_1\psi(p, f_2(y)), \\ h_2(f_2(y), 0)=g_1\psi(p, f_2(y)). \end{cases}$$

Since the base and sides of I^3 constitute a retract of I^3, the mapping F can be extended to all of I^3. Let this extension still be denoted by F. On the top edges of I^3, we have the mappings;

$F(x, 0, 1)=g_1f_1(x)$, $F(0, y, 1)=g_1f_2(y)$,
$F(x, 1, 1)=g_1g_2f_1(x)$, $F(1, y, 1)=g_1^2f_2(y)$.

Now we want a mapping $H: I\times I\to X$ such that

$$H(x,0)=\begin{cases} f_1(2x), & 0\leq x\leq \frac{1}{2}, \\ g_1f_2(2x-1), & \frac{1}{2}\leq x\leq 1, \end{cases}$$
$$H(x,1)=\begin{cases} f_2(2x), & 0\leq x\leq \frac{1}{2}, \\ g_2f_1(2x-1), & \frac{1}{2}\leq x\leq 1. \end{cases}$$

Define a map $H: I\times I\to X$ by the equation

$$H(x,t)=\begin{cases} g_1^{-1}F(2(1-t)x, 2tx, 1), & 0\leq x\leq \frac{1}{2}, \\ g_1^{-1}F(1-2t(1-x), t+(1-t)(2x-1), 1), & \frac{1}{2}\leq x\leq 1. \end{cases}$$

Then

$$g_1^{-1}F(2(1-t)x, 2tx, 1)=g_1^{-1}F(1-t, t, 1) \quad \text{when } x=\frac{1}{2},$$

and

$$g_1^{-1}F(1-2t(1-x), t+(1-t)(2x-1), 1)=g_1^{-1}F(1-t, t, 1) \text{ when } x=\frac{1}{2}.$$

Thus H is well-defined and continuous. Furthermore,

$$H(x,0)=\begin{cases} g_1^{-1}F(2x,0,1)=f_1(2x), & 0\leq x\leq \frac{1}{2}, \\ g_1^{-1}F(1,2x-1,1)=g_1 f_2(2x-1), & \frac{1}{2}\leq x\leq 1, \end{cases}$$

$$H(x,1)=\begin{cases} g_1^{-1}F(0,2x,1)=f_2(2x), & 0\leq x\leq \frac{1}{2}, \\ g_1^{-1}F(2x-1,1,1)=g_2 f_1(2x-1), & \frac{1}{2}\leq x\leq 1. \end{cases}$$

Hence H is the required homotopy between $f_1+g_1f_2$ and $f_2+g_2f_1$.

3. COROLLARY. *Let G be a path connected, abelian topological group. Then the fundamental group $\sigma(G, e, G)$ of a transformation group (G, G) is abelian, where e is the identity element in G.*

Proof. Define a map $\phi: G\times G\rightarrow G$ by $\phi(x, y)=x\cdot y$. Since the group multiplication is continuous, ϕ is continuous. And $\phi(e,e)=e$ and $\phi(x,gy)=x\cdot(gy)=(xg)\cdot y=(gx)\cdot y=\phi(gx,y)=(gx)\cdot y=g(x\cdot y)=g\phi(x,y)$ for $g,x,y\in G$. The maps $f': G\rightarrow G$ by $f'(x)=\phi(x,e)=x$ and $f'': G\rightarrow G$ by $f''(x)=\phi(e,x)=x$ are trivially homotopic to the identity map on X by homotopies that leave g fixed for $g\in G$. Thus the transformation group (G, G) has H-structure and the fundamental group $\sigma(G, e, G)$ of a transformation group (G, G) is abelian.

4. LEMMA. *Let (X, G) be a transformation group with X path connected and let x_1 be in X. If the fundamental group $\sigma(X, x_0, G)$ is abelian then for every pair λ and μ of paths from x_0 to x_1, their induced isomorphisms λ_* and μ_* are equal.*

Proof. Let $[f:g]$ be an element of $\sigma(X, x_0, G)$. Since $\sigma(X, x_0, G)$ is abelian, $[f:g]*[\lambda+\mu\rho:e]=[\lambda+\mu\rho:e]*[f:g]$. Thus

$$[f+g\lambda+g\mu\rho:g]=[\lambda+\mu\rho+f:g] \Longrightarrow f+g\lambda+g\mu\rho\sim\lambda+\mu\rho+f$$
$$\Longrightarrow \lambda\rho+f+g\lambda+g\mu\rho\sim\mu\rho+f \Longrightarrow \lambda\rho+f+g\lambda=\mu\rho+f+g\mu$$
$$\Longrightarrow [\lambda\rho+f+g\lambda:g]=[\mu\rho+f+g\mu:g] \Longrightarrow \lambda_*[f:g]=\mu_*[f:g] \Longrightarrow \lambda_*=\mu_*$$

5. THEOREM. *Let (X, G) be a transformation group with X path connected and G abelian. If the transformation group (X, G) has the H-structure with a base point p then for every pair of paths λ and μ from p to x, their induced isomorphisms λ_* and μ_* are same.*

Proof. From theorem 2.2, the fundamental group $\sigma(X, p, G)$ of a transformation group (X, G) is abelian. For every pair λ and μ of paths from p to x, their induced isomorphisms λ_* and μ_* are equal by lemma 2.4.

6. THEOREM. *Let* (X, G) *and* (Y, G) *be transformation groups, and let* $\phi: (X, G) \to (Y, G)$ *be an equivariant map. If* X *is path connected and* $\phi(X_0) = y_0$ *then* im ϕ_* *is a normal subgroup of* $\sigma(Y, y_0, G)$.

Proof. Define a map $\phi_*: \sigma(X, x_0, G) \to \sigma(Y, y_0, G)$ by the equation

$$\phi_*[f: g] = [\phi f: g] \text{ for } [f: g] \in \sigma(X, x_0, G).$$

If $f \sim f'$, then $\phi f \sim \phi f'$. Thus ϕ_* is well-defined. Morever,

$$\begin{aligned}\phi_*([f_1: g_1]*[f_2: g_2]) &= \phi_*[f_1 + g_1 f_2: g_1 g_2] = [\phi f_1 + g_1 \phi f_2: g_1 g_2] \\ &= [\phi f_1: g_1]*[\phi f_2: g_2] = \phi_*[f_1: g_1]*\phi_*[f_2: g_2]\end{aligned}$$

for $[f_1: g_1], [f_2: g_2] \in \sigma(X, x_0, G)$. Hence the image of ϕ_* is subgroup of $\sigma(Y, y_0, G)$. Now let $[f_3: g_3]$ be an element of $\sigma(Y, y_0, G)$ and $[f_4: g_4]$ be an element of im ϕ_*. Then

$$\begin{aligned}[f_3: g_3]*[f_4: g_4]*[f_3: g_3]^{-1} &= [f_3 + g_3 f_4: g_3 g_4]*[g_3^{-1} f_3 \rho: g_3^{-1}] \\ &= [f_3 + g_3 f_4 + g_3 g_4 g_3^{-1} f_3 \rho: g_3 g_4 g_3^{-1}]\end{aligned}$$

is an element of im ϕ_* since X is path connected.

Reference

1. F. Rhodes, *On the fundamental group of a transformation group*, Proc. London Math. Soc. (3) **16**(1966), 635-650.

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