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A REMARK ON THE KRULL DIMENSION

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1. Introduction

Let A be a commutative local ring and $\mathcal{C}_A$ the category of finite A-modules. Then $d:\mathcal{C}_A\to\mathbf{N}$ defined by $d(E)$, Krull dimension of E, satisfies the following properties:

(1) $\dim(A/\mathfrak{M})=0$

(2) If $0\to E'\to E\to E''\to 0$ is an exact sequence, then
$$d(E)=\text{Max}\ (d(E'),d(E'')).$$

(3) If $0\to E\xrightarrow{a} E\to E/aE\to 0$ where $a\in\mathfrak{M}$ is an exact sequence then
$$d(E)=1+d(E/aE).$$

A main purpose of this note is to show that the above three properties do characterize dimension, i.e., $d:\mathcal{C}_A\to N$ with the above three properties is unique. For the sake of readers, we also give a proof of above properties for Krull dimension based on the notion of Hilbert-Samuel polynomial.

2. Definitions and preliminaries

Let A be a noetherian local ring with maximal ideal $\mathfrak{M}$, E a finite A-module, (E_n) a stable $\mathfrak{M}$-filteration of E. Let
$$\text{gr}(A)=\bigoplus_{\nu=0}^{\infty}\mathfrak{M}^{\nu}/\mathfrak{M}^{\nu+1},\quad \text{gr}(E)=\bigoplus_{\nu=0}^{\infty}\mathfrak{M}^{\nu}E/\mathfrak{M}^{\nu+1}E,\quad \text{gr}_{\nu}(A)=\mathfrak{M}^{\nu}/\mathfrak{M}^{\nu+1},$$
then $\text{gr}_0(A)=A/\mathfrak{M}$ is a field and hence $\text{gr}(A)$ is a noetherian ring, and $\text{gr}(E)$ is a finite $\text{gr}(A)$-module. $\text{gr}_{\nu}(E)=\mathfrak{M}^{\nu}E/\mathfrak{M}^{\nu+1}E$ is a noetherian A-module annhilated by $\mathfrak{M}$.
If $\{x_1, x_2, \cdots\cdots, x_\rho\}$ generates $\mathfrak{M}$, the image $\bar{x}_i$ of the $x_i\in\mathfrak{M}$ generate $\text{gr}(A)$ as an $A/\mathfrak{M}$-algebra and $\bar{x}_i$ has degree 1.

PROPOSITION 1. *$P_E(t)=\sum_{\nu=0}^{\infty}C_{\nu}t^{\nu}$, where $C_{\nu}=[\mathfrak{M}^{\nu}E/\mathfrak{M}^{\nu+1}E : A/\mathfrak{M}]$ is of the form $f(t)/(1-t)$ for some $f(t)\in\mathbf{Z}[t]$.*

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Proof. We shall prove by the induction on ρ, the number of generators of $\mathrm{gr}(A)$ over $A/\mathfrak{M}$. Let $\rho=0$. Then $\mathfrak{M}^n/\mathfrak{M}^{n+1}=0$ for all $n>0$, so that $\mathrm{gr}(A)=A/\mathfrak{M}$ and $\mathrm{gr}(E)$ is a finitely generated $A/\mathfrak{M}$-vector space, and hence $\mathfrak{M}^nE/\mathfrak{M}^{n+1}E=0$ for all $n\gg 0$. Thus $P_E(t)$ is a polynomial.

Suppose $\rho>0$ and the proposition true for $\rho-1$. Multiplication by $\bar{x}_\rho$ is an A-module homomorphism of $\mathfrak{M}^nE/\mathfrak{M}^{n+1}E$ into $\mathfrak{M}^{n+1}E/\mathfrak{M}^{n+2}E$ and hence it gives an exact sequence:

$$0\to P_n/P_{n+1}\to \mathfrak{M}^nE/\mathfrak{M}^{n+1}E\to \mathfrak{M}^{n+1}E/\mathfrak{M}^{n+2}E\to Q_{n+1}/Q_{n+2}\to 0 \qquad *1$$

Let $P=\bigoplus_{\nu=0}^{\infty} P_\nu/P_{\nu+1}$, $Q=\bigoplus_{\nu=0}^{\infty} Q_{\nu+1}/Q_{\nu+2}$. These are both finitely generated A-modules and both annhilated by $\bar{x}_\rho$, hence they are $A/\mathfrak{M}[\bar{x}_1, \bar{x}_2, \cdots\cdots, \bar{x}_\rho]$-module. Applying an additive function to $*1$ we get

$$\lambda(P_n/P_{n+1})-\lambda(\mathfrak{M}^nE/\mathfrak{M}^{n+1}E)+\lambda(\mathfrak{M}E/\mathfrak{M}E)-\lambda(Q_{n+1}/Q_{n+2})=0.$$

Multiplying by t^{n+1} and summing with respect to n we get $(1-t)P_E(t)=P_Q(t)-tP_P(t)+h(t)$, where $h(t)$ is a polynomial. By the induction assumption $P_Q(t)$ and $P_P(t)$ are rational function of the form $g(t)/(1-t)$, and hence

$$(1-t)P_E(t)=P_Q(t)-tP_P(t)+h(t)=f(t)/(1-t)^\rho.$$

Therefore, $P_E(t)=f(t)/(1-t)^{\rho+1}$.

COROLLARY. *For all $n\gg 0$, $\lambda(\mathfrak{M}^nE/\mathfrak{M}^{n+1}E)$ is a polynomial in n of degree $\rho-1$.*

Proof. By the above proposition $\lambda(\mathfrak{M}^nE/\mathfrak{M}^{n+1}E)$ is the coefficient of t^n in $f(t)/(1-t)^\rho$. Suppose $f(1)\neq 0$ and $f(t)=\sum_{k=0}^{N} a_k t^k$. Since

$$(1-t)^{-\rho}=\sum_{k=0}^{\infty}\binom{\rho+k-1}{\rho-1}t^k,\quad \lambda(\mathfrak{M}^nE/\mathfrak{M}^{n+1}E)=\sum_{k=0}^{N} a_k\binom{\rho+n+k-1}{\rho-1}\ \text{for all } n\geq N.$$

Therefore $\mathcal{P}\mathrm{gr}^{(n)}(E)=\lambda(\mathfrak{M}^nE/\mathfrak{M}^{n+1}E)$ is a polynomial in n of degree $\leq\rho-1$. It follows that the function

$$g_E(n)=\lambda_m^E(n)=\lambda(E/\mathfrak{M}^nE)=\sum_{i=0}^{n-1}\lambda(\mathfrak{M}^iE/m^{i+1}E)$$

is also a polynomial in n of degree $\leq\rho$ for all $n\gg 0$. This $g_E(n)$ is the Hilbert-Samuel polynomial in n of E with respect to $\mathfrak{M}$. We shall let $\deg g_E(n)=d(E)$.

REMARK. Define $\delta(E)$ to be the order of the pole at 1 in $P_E(t)$. As we know easily $\delta(E)=d(E)-1$. In fact,

$$\lambda(E/\mathfrak{M}^{n+1}E)=\lambda(E/\mathfrak{M}E)+(\mathfrak{M}E/\mathfrak{M}^2E)+\cdots\cdots+(\mathfrak{M}^nE/\mathfrak{M}^{n+1}E).$$

Put $P_E(t)=\sum_{\nu=0}^{\infty}C_\nu t^\nu$, where $C_\nu=\lambda(\mathfrak{M}^\nu E/\mathfrak{M}^{\nu+1}E)$. Then since $C_\nu=\lambda_\nu-\lambda_{\nu-1}$, where $\lambda_\nu=C_1+C_2+\cdots\cdots+C_\nu$, we have

$$P_E(t) = \sum_{\nu=0}^{\infty} (\lambda(E/\mathfrak{M}^{\nu+1}E) - \lambda(E/\mathfrak{M}^{\nu}E))t^{\nu}$$
$$= \sum \lambda(E/\mathfrak{M}^{\nu+1}E)t^{\nu} - (\sum (\lambda(E/\mathfrak{M}^{\nu}E)t^{\nu-1})t = (1-t)\sum \lambda(E/\mathfrak{M}^{\nu+1}E)t^{\nu}.$$

Therefore,

$$\sum \lambda(E/\mathfrak{M}^{\nu+1}E)t^{\nu} = \frac{1}{1-t}P_E(t) = \frac{1}{1-t}\sum \lambda(\mathfrak{M}^{\nu}E/\mathfrak{M}^{\nu+1}E)t^{\nu}.$$

As we expect, $\delta(E) = d(E) - 1$.

PROPOSITION 2. *Let* A, $\mathfrak{M}$, E *be as in Proposition* 1 *and* $\mathcal{C}_A$ *the category of finite A-modules. For any objects* $E, E', E'' \in \mathcal{C}_A$, *if*

$$0 \to E' \to E \to E'' \to 0$$

is an exact sequence of finite A-module, then $d(E) = \text{Max}\ (d(E'), d(E''))$

Proof. From given exact sequence we know

$$0 \to E' + \mathfrak{M}^n E/\mathfrak{M}^n E \to E/\mathfrak{M}^n E \to E''/\mathfrak{M}^n E'' \to 0$$

is an exact sequence of finite A-module, and so $E/E' + \mathfrak{M}^n E \simeq E''/\mathfrak{M}^n E''$. Since

$$\lambda(E''/\mathfrak{M}^n E'') = \lambda(E/E' + \mathfrak{M}^n E) \leq \lambda(E/\mathfrak{M}^n E),$$

we get $d(E'') \leq d(E)$. Furthermore,

$$\lambda_m^E(n) - \lambda_m^{E''}(n) = \lambda(E/\mathfrak{M}^n E) - \lambda(E''/\mathfrak{M}^n E'')$$
$$= \lambda(E/\mathfrak{M}^n E) - \lambda(E/E' + \mathfrak{M}^n E) = \lambda(E' + \mathfrak{M}^n E/\mathfrak{M}^n E)$$
$$= \lambda(E'/\mathfrak{M}' \cap \mathfrak{M}^n E),$$

and there exists $r > 0$ such that $E' \cap \mathfrak{M}^n E \subset \mathfrak{M}^{n-r} E'$ for all $n > r$ by Artin-Rees. Thus

$$\lambda(E'/\mathfrak{M}^n E') \geq \lambda(E'/E' \cap \mathfrak{M}^n E) \geq \lambda(E'/\mathfrak{M}^{n-r} E').$$

This means that $\lambda_m^E(n) - \lambda_m^{E''}(n)$ and $\lambda_m^{E'}(n)$ have the same degree and the same leading term.

PROPOSITION 3. *Let* A, $\mathfrak{M}$, E *be as in proposition* 1, *and* $a \in \mathfrak{M}$ *non-zero diviser on* E. *Then* $d(E) - 1 = d(E/aE)$.

Proof. From the given condition, we get an exact sequence:

$$0 \to aE \to E \to E/aE \to 0$$

and hence

$$0 \to aE + \mathfrak{M}^n E/\mathfrak{M}^n E \to E/\mathfrak{M}^n E \to E/aE/\mathfrak{M}^n(E/aE) \to 0$$

and then

$$\lambda_m^{E/aE}(n) = \lambda(E/aE + \mathfrak{M}^n E) = \lambda(E/\mathfrak{M}^n E) - \lambda(aE + \mathfrak{M}^n E/\mathfrak{M}^n E),$$
$$aE + \mathfrak{M}^n E/\mathfrak{M}^n E \cong aE/aE \cap \mathfrak{M}^n E \cong E/(\mathfrak{M}^n E : a) \text{ and } \mathfrak{M}^{n-1}E \subseteq (\mathfrak{M}^n E : a).$$

Hence

$$\lambda_m^{E/aE}(n) \geq (E/\mathfrak{M}^n E) - \lambda(E/\mathfrak{M}^{n-1}E) = \lambda_m^E(n) - \lambda_m^E(n-1).$$

It follows that $d(E/aE) \geq d(E)-1$. On the other hand, $aE \cong E$ as A-modules by the hypothesis on a. We have an exact sequence:

$$0 \to aE/aE \cap \mathfrak{M}^n E \to E/\mathfrak{M}^n E \to E/aE/\mathfrak{M}^n(E/aE) \to 0.$$

Hence

$$\lambda(aE/aE \cap \mathfrak{M}^n E) - \lambda(E/\mathfrak{M}^n E) + \lambda(E/aE/\mathfrak{M}^n(E/aE)) = 0$$

for all $n \gg 0$. By the Artin-Rees, $aE \cap \mathfrak{M}^n E$ is a stable $\mathfrak{M}$-filteration of E.

Since $aE \cong E$, $\lambda(E/aE \cap \mathfrak{M}^n E)$ and $\lambda_m^E(n)$ have the same leading term because the degree and leading coefficient of Hilbert-Samuel polynomial depend only on E and m. not on the filteration chosen.

Therefore $d(E/aE) \leq d(E)-1$.

3. Main theorem

THEOREM. *Let A be a local noetherian ring with maximal ideal $\mathfrak{M}$, E a finite A-module,* $\mathrm{gr}(E) = \bigoplus_{\nu=0}^{\infty} \mathfrak{M}^\nu E/\mathfrak{M}^{\nu+1}E$, $P_E(t) = \sum_{\nu=0}^{\infty} C_\nu t^\nu$, *where* $C_\nu = [\mathfrak{M}^\nu E/\mathfrak{M}^{\nu+1}E : A/\mathfrak{M}]$, $\delta(E)$ *the order of the pole at* 1 *in* $P_E(t)$ *and* $\mathcal{C}_A$ *the category of finite A-modules.*

Define $E \to \lambda(E) \in \mathbf{N}$ *non negative, then following hold:*

(1) $\mathfrak{M}^n E = 0$ *for some* $n > 0 \Longleftrightarrow$ (1′) $\delta(E) = \delta(A/\mathfrak{M}) = 0$

(2) $0 \to E' \to E \to E'' \to 0$ *is exact and* $\delta(E) = \mathrm{Max}(\delta(E'), \delta(E''))$

(3) $0 \to aE \to E \to E/aE \to 0$ *is exact, where* $a \in \mathfrak{M}$

$\Longrightarrow \delta(E/aE) = \delta(E) - 1.$

Conversely, this map is uniquely determined by the above conditjons.

Proof. (1) $\mathfrak{M}^n E = 0$ for some n implies $\mathfrak{M}(\mathfrak{M}^n E) = 0$, and hence $\mathfrak{M}^{n+1}E = 0$. Since $\mathfrak{M}^{n+k}E = 0$, for all $k \geq 0$, $\mathfrak{M}^{n+k}/\mathfrak{M}^{n+k+1}E = 0$.

Thus $C_{n+k} = \dim_{A/m}(\mathfrak{M}^{n+k}E/\mathfrak{M}^{n+k+1}) = 0$. Therefore, $P_E(t) = \sum_{\nu=0}^{\infty} C_\nu t^\nu = \sum_{\nu=0}^{n} C_\nu t^\nu$ because $C_\nu = 0$ if $\nu > n$, so the order of the pole is zero, i. e., $\delta(E) = 0$.

(1) $\Longleftrightarrow$ (1′) From the given condition, we get a chain $E \supset E_1 \supset \cdots \supset E_r = 0$ of submodules such that $E_i/E_{i+1} \simeq A/P_i$ by J. P. Serre.

Since $P_i \supseteq \mathfrak{M} \Longrightarrow P_i \supseteq \mathfrak{M}$, $P_i = \mathfrak{M}$. Hence $\delta(E) = \delta(A/P) = 0$.

(2) & (3) followes from Proposition 2, 3 and Remark.

Conversely, by J. P. Serre, there exists a chain such that $E = E_0 \supset E_1 \supset \cdots \supset E_r = 0$, where $E_i/E_{i+1} \cong A/Pi$ for some i.

From

$$0 \to E_1 \to E \to E/E_1 \to 0,$$
$$0 \to E_2 \to E_1 \to E_1/E_2 \to 0,$$

....

we know

$$d(E) = \mathrm{Max}\{d(E/E_1),\ d(E_1/E_2), \cdots\cdots\cdots d(E_{r-1}/E_r)\}$$
$$= \mathrm{Max}\ \{d(A/P_1), d(A/P_2), \cdots\cdots\cdots\cdots d(A/P_r)\}.$$

If $d(A/P) = \dim(A/P)$, then $d(E) = \dim(E)$. Thus if $d(E) \neq \dim(E)$ for some module E then $d(A/P) \neq \dim(A/P)$ for some prime ideal P.

Assume that $d(E) \neq \dim(E)$, and then choose a maximal one among all pime ideals P for which $d(A/P) \neq \dim(A/P)$.

Let P be a maximal one.

(a) $P \neq \mathfrak{M}$ because if $P = \mathfrak{M}$ then $d(A/P) = 0$ by (1′) whereas $\dim(A/\mathfrak{M}) = 0$.

(b) Since $P \subsetneqq \mathfrak{M}$ we can pick $a \in \mathfrak{M} - P$. The multiplication by a in A/P is one-one, i.e., $0 \to A/P \to A/P \to A/(P+aA) \to 0$ is exact. Then, by (3), we have

$$d(A/P+aA) = d(A/P) - 1,\ \text{ i.e., }\ d(A/P) = 1 + d(A/P+aA).$$

However choose $A/P+aA = E \supset E_1 \supset \ldots \supset E_r = 0$ such that $E_i/E_{i+1} \cong A/P_i$, then $P_i \supset P + aA \supseteqq P$. Because P was a maixmal amongest $d(A/P) \neq \dim(A/P)$ we must have $d(A/P_i) = \dim(A/P_i)$ for all i. Therefore,

$$\mathrm{Max}(d(E/E_1), d(E/E_2), \cdots\cdots) = d(A/P+aA)$$
$$= \mathrm{Max}(d(A/P_1), d(A/P_2), d(A/P_3), \cdots\cdots) = \dim(A/P+aA).$$

Hence

$$d(A/P) = 1 + d(A/P+aA) = 1 + \dim(A/P+aA) = \dim(A/P),$$

which is a contradiction.

COROLLARY *Let $A \to B$ be a local map of noethorian local rings, and E a finite B-module which is A-flat. Then for any finite A-module M we have*

$$\dim_A(M) = \dim_B(M \otimes_A E)\ -\ \dim_B(E/\mathfrak{M}E).$$

Proof. Let $\mathscr{C}_A \to N$, where $\delta(M) = \dim_B(M \otimes_A E) - \dim_B(E/\mathfrak{M}E)$. Then

(i) $\delta(A/\mathfrak{M}) = \dim_B(A\mathfrak{M} \otimes E) - \dim_B(E/\mathfrak{M}E) = \dim_B(E/\mathfrak{M}E) - \dim_B(E/\mathfrak{M}E) = 0$.

(ii) $0 \to M' \to M \to M'' \to 0$ is exact and

$0 \to M' \otimes_A E \to M \otimes_A E \to M'' \otimes_A E \to 0$ is exact since E is A-flat.

So $\dim_B(M \otimes_A E) = \mathrm{Max}(\dim_B(M' \otimes_A E),\ \dim_B(M'' \otimes_A E))$. Therefore,

$$\delta(M) = \mathrm{Max}(\delta(M'), \delta(M'')).$$

(iii) $0 \to M \to M \to M/aM \to 0$, where $a \in \mathfrak{M}$, is exact.

$\Longrightarrow 0 \to M \otimes_A E \to M \otimes_A E \to M/aM \otimes_A E \to 0$ is exact.

$\Longrightarrow 0 \to M \otimes_A E \to M \otimes_A E \to M \otimes_A E/a(M \otimes_A E) \to 0$ is exact.

$\Longrightarrow \dim_B(M \otimes_A E) = 1 + \dim_B(M \otimes_A E/a(M \otimes_A E))$.

So $\delta(M) = 1 + (M/aM)$.

By uniqueness,

$$\dim_A(M) = \dim_B(M \otimes_A E) - \dim_B(E/\mathfrak{M}E).$$

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