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BIRATIONAL INVARIANCE OF COHOMOLOGICAL BRAUER GROUP

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Let X be an algebraic scheme, and let $Br(X)$ denote the Brauer group of equivalence classes of Azumaya O_X-algebras. Grothendieck showed that there is an injection

$$Br(X) \to H_{et}^2(X, Gm),$$

which is not necessarily a surjection. However, $Br(X) \cong H_{et}^2(X, Gm)$ if (dimension X) $\leqslant 1$ or X is regular with dimension $\leqslant 2$ [1]. In particular, the Brauer group of complete nonsingular algebraic surfaces over a separably closed field is birationally invariant. Let $Br'(X) = H^2(X, Gm)$ denote the cohomological Brauer group of X.

The present note concerns with a certain algebraic surface over a separably closed field with only finitely many isolated singular points. Then we show Br', the cohomological Brauer group, is birationally invariant as well.

THEOREM *Let X be a normal surface over a separably closed field k with finitely many isolated singular points, and let $\pi : \tilde{X} \to X$ be a desingularization. Then there is an exact sequence:*

$$0 \to Pic\, X \to Pic\, \tilde{X} \to \prod_{x \in S} Pic(\tilde{X}(x)) \to H^2(X, Gm) \to Br(\tilde{X}) \to 0,$$

where S is a set of singular points and Gm denotes the sheaf (in the etale topology) of multiplicative groups.

Proof. The Leray spectral sequence

$$E_2^{pq} = H^p(X, R^q\pi_* Gm) \implies E^{pq} = H^{p+q}(\tilde{X}, Gm)$$

yields an exact sequence of low term degree

$$0 \to H^1(X, R^0\pi_* Gm) \to H^1(\tilde{X}, Gm) \to H^0(X, R^1\pi_* Gm) \to H^2(X, R^0\pi_* Gm)$$
$$\to H^2(\tilde{X}, Gm) \to H^1(X, R^1\pi_* Gm).$$

For each closed point $x \in X$, let $X(x) = Spec(O_{X,x}^{sh})$ and $\tilde{X}(x) = \tilde{X}_X \times X(x)$, where $O_{X,x}^{sh}$ denotes the strict henselization of $O_{X,x}$. By Satz 6.4.1, [3] we have,

$$(R^q\pi_* Gm)_x = H^q(\tilde{X}(x), Gm)$$

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for each $q \geqslant 1$. Then since π is a product of quadratic transformations

$$R^0\pi_* Gm = Gm.$$

Since $X - S \cong \tilde{X} - \pi^{-1}(S)$, if x is a regular point then $\tilde{X}(x) = X(x)$ and $H^1(\tilde{X}(x), Gm) = 0$. Therefore we get

$$R^1\pi_* Gm = \prod_{x \in S} (R^1\pi_* Gm)_x.$$

Now let P_x be a geometrical point of X with a structure morphism $i : P_x \to X$, i. e., for a closed point x in X,

$$P_x = Spec(k(x)) = Spec(k).$$

Then $(R^1\pi_* Gm)_x = i_*(R^1\pi_* Gm)_{P_x}$ and hence

$$P^1\pi_* Gm = \prod_{x \in S} i_*(R^1\pi_* Gm)_{P_x}.$$

Using this:

$$\begin{aligned} H^0(X, R^1\pi_* Gm) &\cong \prod_{x \in S} H^0(P_x, i_*(R^1\pi_* Gm)_{P_x}) \\ &\cong \prod_{x \in S} i_*(R^1\pi_* Gm)_{P_x} && \text{(see [3], Lemma 6. 2. 3)} \\ &\cong \prod_{x \in S} H^1(\tilde{X}(x), Gm) && \text{(see [3], Prop. 6. 2. 2)} \\ &\cong \prod_{x \in S} Pic(\tilde{X}(x)) \\ H^1(X, R^1\pi_* Gm) &\cong \prod_{x \in S} H^1(P_x, i_*(R^1\pi_* Gm)_{P_x}) = 0. \end{aligned}$$

Since P_x is a geometrical point, the last equality follows from the equivalence of the etale cohomology and Galois cohomology over a field and Hilbert theorem 90.

Since $\tilde{X}$ is a regular surface, $H^2(\tilde{X}, Gm) = Br(\tilde{X})$ and so the theorem follows.

Now let X be a surface which satisfies the conditions in the theorem and the following property (*):

(*) For each $x \in S$, $Cl(O_{X,x}) = Cl(O_{X,x}{}^{sh})$.

Geometrically factorial rings satisfy the above property. For other examples of such rings see [2].

COROLLARY *Let X be as above and let $\tilde{X}$ be the desingularization. Then*

$$Br'(X) \cong Br(\tilde{X}).$$

Proof. Let K be the rational functional function field on X. Then K is also the rational function field on $\tilde{X}$ and we have

$$ker(Br'(X) \to Br(\tilde{X})) = ker(Br'(X) \to (K))$$

since $\tilde{X}$ is regular, $Br'(\tilde{X}) \to Br(K)$ is an injection.

Now the property (*) insures the injectivity of $Br'(X) \to Br(K)$ (see [1], Remark 1. 11. b), and the corollary follows from the exact sequence in the theorem.

REMARK If X and Y are birationally equivalent surfaces with finitely many isolated singular points, they have birationally equivalent desingularizations. Hence from the corollary and the remark at the beginning, the Brauer group of complete surfaces over a separably closed field having only finitely many isolated singular points with the property (*) is birationally invariant.

References

1. A. Grothendieck, *Le Groupe de Brauer I, II, in Dix Exposés sur la cohomologie des Schema*, North-Holland (1968).
2. H. Lee, *The Brauer group of some normal domains* (preprint).
3. G. Tamme, *Einführung in die étale Kohomologie*, Vorlesung, Math. Institut der Univ. Göttingen (1976).

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