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SPECTRA OF DECOMPOSABLE OPERATORS

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1. Introductien

The well-known Weyl-von Neumann theorem about compact pertubations says that for two normal operators in a separable infinite dimensional Hilbert space they are unitary equivalent up to compacts if and only if they have the same essential spectrum. That is, the essential spectrum for normal operators (i. e., eigenvalues with infinite multiplicity plus limit points in the spectrum) is unitary invariant modulo compact ideals. Recently the Weyl-von Neumann theorem has been generalized into the context of separably acting von Neumann algebras of type II by Kaftal and author [3]. In this paper somewhat similar programme is undertaken.

Let $T_1=\int_{\Lambda}^{\oplus} T_1(\lambda)\,d\mu$ and $T_2=\int_{\Lambda}^{\oplus} T_2(\lambda)\,d\mu$ be decomposable operators on $H=\int_{\Lambda}^{\oplus} H(\lambda)\,d\mu$ of separable Hilbert spaces. We study the following problem: if the spectrum of $T_1(\lambda)$ and $T_2(\lambda)$ is identical almost everywhere, then does one get the same spectrum of T_1 and T_2? In general, this is not the case. However, for normal decomposable operators it turns out that the spectrum of $T_1(\lambda)$ and $T_2(\lambda)$ are same almost everywhere if and only if for each central projection E the spectrum of T_1E and T_2E are same. Our method to attack this problem is closely related to [3]. We mention that several authors [1, 2, 4] used direct integral theory to investigate individual operators.

2. Preliminaries

We now consider a few of basics of direct integral decomposition theory. Our discussion will be based on Schwartz [5].

Let H_0 be a fixed separable infinite-dimensional Hilbert space and μ be a finite positive regular measure defined on the Borel subsets of a separable metric space A. Consider functions defined on A with values in H_0. Such a function $f(\lambda)$ will be called *measurable* if the complex-valued function

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$(f(\lambda), x)$ is measurable in the usual sense for all x in H_0 $\{(\ ,\ )$ denotes the inner product in $H_0\}$. For each λ in Λ, we set $H(\lambda)=H_0$. Then the symbol

$$H=\int_{\Lambda}^{\oplus} H(\lambda)\,d\mu$$

denotes the set all equivalence classes of measurable functions defined on Λ ($f(\lambda)$ and $g(\lambda)$ are equivalent if $f(\lambda)=g(\lambda)$ almost everywhere) such that

1. $f(\lambda)$ is in $H(\lambda)$ for all λ in Λ
2. $\int_{\Lambda} \|f(\lambda)\|^2 d\mu<\infty$

one defines an inner product on H by $(f,g)=\int_{\Lambda} (f(\lambda), g(\lambda))\,d\mu$; this makes H into a complete separable Hilbert space called a *direct integral Hilbert space.* Let $T(\lambda)$ be a function defined on Λ with values in the set $L(H(\lambda))$ of operators (operator will mean bounded linear transformation) on $H(\lambda)$ for each λ. Then the operator-valued function $T(\lambda)$ is said to be *measurable* if $T(\lambda)f(\lambda)$ is measurable for all f in H. Then the function $f(\lambda)\to T(\lambda)f(\lambda)$ defines a bounded operator on H with norm ess. $\sup\|T(\lambda)\|$, called the *direct integral* of $T(\lambda)$ and written

$$T=\int_{\Lambda}^{\oplus} T(\lambda)\,d\mu$$

An operator on a direct integral Hilbert space H which has the above form is said to be *decomposable.* One can topologize the unit ball, denoted by B_1, of $L(H)$ by means of uniform, strong*, strong and weak convergences of operators. In each of the last topologies, there is a metric in which the unit ball is separable and complete (see [5, pp. 38–39]). Furthermore, although these last three topologies are different, the Borel structures generated by each topology are identical. This enables us to choose any one of the last three topologies as far as the Borel structure is concerned. Hence a decomposable operator $T(\lambda)$ can be regared as a Borel map from Λ to a bounded subset of $L(H_0)$. Finally, $\sigma(T)$ will denote the spectrum of T.

3. Spectra of decomposable operators

The following is well-known.

3. 1 LEMMA. *Let* $T=\int_{\Lambda}^{\oplus} T(\lambda)\,d\mu$ *be a decomposable operator on* $H=\int_{\Lambda}^{\oplus} H(\lambda)\,d\mu$ *Then* $\sigma(T(\lambda))$ *is contained in* $\sigma(T)$ a. e.

Proof: Suppose that a is not in the spectrum of T. Then $T-a$ is invertible and its inverse should be $\int_{\Lambda}^{\oplus}(T(\lambda)-a)^{-1}d\mu$. Thus $T(\lambda)-a$ is invertible almost everywhere. This completes the proof.

3. 2. Suppose that we are given two operators T_1 and T_2 with $\sigma(T_1(\lambda))=\sigma(T_2(\lambda))$ a. e. Our question is whether $\sigma(T_1)=\sigma(T_2)$ or not. The answer is negative in general. Consider the following examples. Let $\Lambda=\mathbf{Z}^+$ with discrete topology and μ be the counting measure. For each natural number n, let $A(n)$ be the weighted shift with weights $(1, 1, \cdots\cdots, 1, 0, \cdots\cdots)$ (1 appears n times) and $B(n)$ the weighted shift with weights $(1/2,\ 1/2, \cdots\cdots, 1/2, 0, 0, \cdots)$ (1/2 appears n times). Then both of them are nilpotent operators and hence $\sigma(A(n))=\sigma(B(n))=\{0\}$. But the spectrum of the operator $A=\Sigma^{\oplus} A(n)$ is the unit disc in the complex plane while that of $B=\Sigma^{\oplus}B(n)$ is the disc of radius 1/2 with center at origin [6, p. 66]. Also, these examples show that the union of spectra of almost all of $T(\lambda)$ is properly contained in the spectrum of T. When does one get the equality of two sets?

Suppose that a is not in $\sigma(T(\lambda))$ a. e. Then the inverse of $T(\lambda)-a$ exists a. e. Hence one would attempt to integrate them to get linear transformation $\int_{\Lambda}^{\oplus}(T(\lambda)-a)^{-1}d\mu$ but this won't be operator unless $\|(T(\lambda)-a)^{-1}\|$ is essentially bounded. If it is essentially bounded, then the above is indeed the direct integral of $(T(\lambda)-a)^{-1}$ and hence it is the inverse of $T-a$. One way to assure that $\|T(\lambda)-a)^{-1}\|$ is essentially bounded the following concepts.

We say that the resolvent, $R(T;a)$, of T has the growth rate of order α with constant k if $\|R(T;a)\|\leq k/d(a,\sigma(T))^{\alpha}$, where d denotes the dis tance between a and $\sigma(T)$. Note that any normal operator N has the growth rate of order 1 with constant 1. For, spectral theorem asserts that the C^*–subalgebra generated by N and the identity operator I is *–isomorphic to the complex-valued continuous functions on $\sigma(N)$ with N corresponding to the function $id(z)$ for z in $\sigma(N)$. For a not in $\sigma(N)$, the function $f(z)=1/(id(z)-a))$ is continuous with norm $\|f(z)\|=1/d(a,\sigma(N))$ and it is the inverse of $id(z)-a$. Hence the resolvent $R(N;a)$ satisfies the same condition, i. e., $\|R(N;a)\|=1/d(a,\sigma(N))$. The following two propositions are taken from [4]. For the reader's convenience the complete proofs are given.

3. 3. PROPOSITION *Let* $T=\int_{\Lambda}^{\oplus}T(\lambda)d\mu$. *If* $T(\lambda)$ *has the uniform growth rate of order a* 1 *with constant* k a. e. *then, for* $a\notin\overline{\bigcup_{\lambda\notin\Omega}\sigma(T(\lambda))}=S$ *with* Ω *of measure zero,* $T-a$ *is invertible.*

Proof: By previous discussion, it suffices to show that, for such an a, $\|(T(\lambda)-a^{-1}\|$ is essentially bounded. Since

$$\|(T(\lambda)-a)^{-1}\|\leq k/d(a,\sigma(T(\lambda)))^{\alpha}\leq k/d(a,S)^{\alpha},$$

it is essentially bounded. This completes the proof.

3.4. PROPOSITION *Let* T, $T(\lambda)$, α *and* k *be same as in Proposition* 3.3. *Then there exists a set* Ω *of measure zero suct that* $\sigma(T)=\overline{\bigcup_{\lambda\notin\Omega}\sigma(T(\alpha))}$

Proof: It is obvious form the previous proposotion.

4. Decomposable normal operators

4.1 THEOREM *Let* $N_1=\int_{\Lambda}^{\oplus}N_1(\lambda)d\mu$ *and* $N_2=\int_{\Lambda}^{\oplus}N_2(\lambda)d\mu$ *be normals. If* $\sigma(N_1(\lambda))=\sigma(N_2(\lambda))$ a. e., *then* $\sigma(N_1E)=\sigma(N_2E)$ *for all central projection* E.

Proof; Since both $N_1(\lambda)$ and $N_2(\lambda)$ are normal a. e., they have the same growth rate of order 1. Hence Proposition 3.4 applies to N_1E and N_2E for all central projection E.

4.2. Before we state and prove the converse of Theorem 4.1, we need more about measure-theoretic nature of the direct integral theory. Let $N=\int_{\Lambda}^{\oplus}N(\lambda)d\mu$ be a fixed normal operator. Without loss of generality we can assume that N (and hence $N(\lambda)$) is a contraction. Let D denote the unit disc in the complex plane. Define $\phi_N : \Lambda\times D\to\Lambda\times B_2$ by

$$\phi_N(\lambda,r)=(\lambda,N(\lambda)-r).$$

Then ϕ_N is a Borel map. Hence the sets $\{(\lambda,r)\mid N(\lambda)-r$ is invertible$\}$ and $\{(\lambda,r)\mid N(\lambda)-r$ is not invertible$\}$ are Borel sets in $\Lambda\times D$. Now let $N_1=\int_{\Lambda}^{\oplus}N_1(\lambda)d\mu$ and $N_2=\int_{\Lambda}^{\oplus}N_2(\lambda)d\mu$ be two normals. Set

$$A=\{(\lambda,r)\mid r\notin\sigma(N_1(\lambda))\}$$

and

$$B=\{(\lambda,r)\mid r\in\sigma(N_2(\lambda))\}$$

Then two sets are Borel, so is their intersection. Let

$$\Omega=\{\lambda\in\Lambda\mid\{\lambda\}\times D\cap A\cap B\neq\phi\}$$

Then by "the principle of measurable choice" (see [5, p. 35]), there exist Borel sets Ω_1 and Ω_2 of Ω and a measurable function ϕ on Ω_1 such that $\mu(\Omega_2)=0$, $\Omega_1\cup\Omega_2\supset\Omega$ and the graph of ϕ is contained in $A\cap B$. By changing the role of N_1 and N_2, it is enough to show that $\mu(\Omega_1)=0$ to get the converse of Theorem 4.1. The following is a key step toward our goal. In general, spectrum does not behave nicely with the norm topology, but with commutative algebras they do.

4.3 LEMMA *Let X and Y be compact Hausdorff spaces and let $f_n, f \in C(X)$ and g_n, $g \in C(Y)$. If f_n and g_n converge to f and g in norm, respectively, and if $\sigma(f_n)=\sigma(g_n)$ for all n, then $\sigma(f)=\sigma(g)$.*

Proof. Suppose $r \in \sigma(f)$, i.e., there exists a point x_0 in X such that $f(x_0)=r$. Then for each n there exists a point y_n in Y such that $g(y_n)=f_n(x_0)$. Since Y is compact Hausdorff, there exists a convergent subsequence $\{y_{n_k}\}$ and let y_0 be its limit point. Since $\{g_{n_k}(y_{n_k})\}$ converges to r, so does $\{g_{n_k}(y_0)\}$. But $\{g_{n_k}(y_0)\}$ converges to $g(y_0)$ and hence $g(y_0)=r$. Therefore $\sigma(f) \subset \sigma(g)$. By interchanging the role of f and g, we get $\sigma(f)=\sigma(g)$. This proves the lemma.

4.4 THEOREM *Let $N_1=\int_{\wedge}^{\oplus} N_1(\lambda)\,d\mu$ and $N_2=\int_{\wedge}^{\oplus} N_2(\lambda)\,d\mu$ be two normals. If $\sigma(N_1E)=\sigma(N_2E)$ for all central projection E, then $\sigma(N_1(\lambda))=\sigma(N_2(\lambda))$* a.e.

Proof; Let A, B, ϕ, Ω and Ω_1 be same as in 4.2. It suffices to prove that $\mu(\Omega_1)=0$. Let E_1 be the central cover of Ω_1. Let ϕ_n be simple functions converging to ϕ uniformly. Suppose that $\mu(\Omega_1)\neq 0$. Since for any complex number α, $\sigma((N_1-\alpha)E)=\sigma((N_2-\alpha)E)$, we have $\sigma((N_1-\phi_n)E)=\sigma((N_2-\phi_n)E)$. Since the C^*-subalgebra generated by $N_1E_1-\phi_nE_1$, $N_1E_1-\phi E_1$ and the identity is commutative, by Gelfand theorem it is *-isomorphic to $C(X)$ for some compact Hausdorff space X. We now apply $N_1E_1-\phi_nE_1$, $N_1E_1-\phi E_1$, $N_2E_1-\phi_nE_1$ and $N_2E_1-\phi E_1$ to 4.3 to get $\sigma(N_1E_1-\phi E_1)=\sigma(N_2E_1-\phi E_1)$. We claim that $\operatorname{ess.\,inf}_{\lambda\in\Delta} d(\phi(\lambda), \sigma(N, (\lambda)))=0$ for all measurable subset Δ of Ω_1, where d denotes the distance. For, if

$$\text{ess. inf}\, d(\phi(\lambda),\ \sigma(N_1(\lambda)))>0$$

for some non-zero measurable set Δ, then since

$$(N_1E-\phi E)^{-1}=1/\text{ess. inf}\, d(\phi(\lambda), \sigma(N_1(\lambda)))<\infty,$$

$N_1E-\phi E$ is invertible but $N_2E-\phi E$ is not invertible by choice of ϕ, where E is the central cover of Δ. Hence this would imply that $\sigma(N_1E-\phi E)\neq \sigma(N_2E-\phi E)$, which contradicts to the hypothesis. Hence $\phi(\lambda)\in\sigma(N_1(\lambda))$ a.e. (since $\sigma(N_1(\lambda))$ is compact, and $d(\phi(\lambda), \sigma(N_1(\lambda)))=0$ implies that $\phi(\lambda)\in\sigma(N_1(\lambda))$), which can't be the case unless $\mu(\Omega_1)=0$.
This completes the proof.

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