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NECESSITY OF THE NIRENBERG-TREVES CONDITION

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Let $P=\frac{\partial}{\partial t}+ib(x,t)\frac{\partial}{\partial x}+f(x,t)$ be a linear partial differential operator with two independent variables defined in Ω, an open subset of R^2, where $b(x,t)$ is a real valued C^∞ function and $f(x,t)$ is a complex valued C^∞ function in Ω.

Assume that $b(x,t)$ has a zero at $\omega_0=(x_0,t_0)\in\Omega$ of finite order. In this particular case Nirenberg and Treves proved that if the function $b(x_0,\cdot)$ of t changes sign at $t=t_0$, then the linear partial differential equation $Pu=f$ is not locally solvable at ω_0 for the generic f (cf. [4]). In this paper we shall remove the assumption that $b(x_0,\cdot)$ vanishes at ω_0 *up to finite order* and shall prove the similar result holds without this assumption.

For this purpose we introduce the following condition $(\tilde{P})$; namely, $P=\frac{\partial}{\partial t}+ib(x,t)\frac{\partial}{\partial x}+f(x,t)$ defined in an open set Ω satisfies the condition $(\tilde{P})$ if and only if

$(\tilde{P})$ *There exists a relatively compact subset of* Ω,

$$\Omega^0_\varepsilon=\{(x,t)\mid |x-x_0|<\varepsilon,\ c-\varepsilon<t<d+\varepsilon\}$$

for some $\varepsilon>0$ and $b(x,t)$ *satisfies the condition that* $b(x_0,c)<0<b(x_0,d)$.

The principal aim of this paper is to prove the following

THEOREM 1. *If P satisfies the condition $(\tilde{P})$ in Ω, then there exists $f\in C_0^\infty(\Omega)$ such that for no $u\in\mathcal{D}'(\Omega)$ $Pu=f$; that is, $Pu=f$ is not solvable in Ω for generic f.*

Thus in the Theorem 1, $b(x_0,\cdot)$ may vanish of infinite order and may oscillate in an arbitrary fashion as far as it satisfies the condition $(\tilde{P})$. For the proof, we shall apply the Moyer's technique developed in [3].

1. Preliminaries

The arguments given here to prove the Theorem 1 are extensions of Hörmander's in [2]. Like his they are based on an inequality involving

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the formal transpose operator for the local solvability with the aid of the Baire category theorem. It asserts that tf the equation $Pu=f$ has a solution u which is a distribution in an open set Ω for every $f\in C_0(\Omega)$ and if U is an open set with compact closure in Ω, then there are constants C, k and N such that

(1) $\left|\int f\bar{v}dx\right|\leqq C\sum_{|\beta|\leqq N}\sup|\partial^\alpha f|\sum_{|\alpha|\leqq k}\sup|\partial^\beta {}^*Pv|$ for all f, $v\in C_0^\infty(V)$.

Here ${}^*p$ denotes the formal adjoint of P; i. e., if $P=\sum_{|\alpha|\leqq m}C_\alpha(X)\partial_x{}^\alpha$, then ${}^*Pu=\sum_{|\alpha|\leqq m}(-1)^{|\alpha|}\partial_x{}^\alpha(\bar{C}_\alpha u)$. In our case, since $P=\frac{\partial}{\partial t}+ib(x,t)\frac{\partial}{\partial x}+f(x,t)$,

(2) ${}^*P=-\left(\frac{\partial}{\partial t}-ib(x,t)\frac{\partial}{\partial x}\right)+g(x,t)$ where $g(x,t)=i\frac{\partial b}{\partial x}+\bar{f}$.

Now suppose $P=\frac{\partial}{\partial t}+ib\frac{\partial}{\partial x}+f$ satisfies the condition $(\tilde{P})$ in Ω. Then by the continuity arguments, P satisfies the condition $(\tilde{P}_1)$ in Ω; namely,

$(\tilde{P}_1)$ There exists a relatively compact open subset of Ω,

$$\Omega_\varepsilon=\{(x,t)\mid |x-x_0|<\varepsilon,\ c<t<d\}$$

and $b(x,t)$ satisfies that

$$b(x_0,t_1)\xi_1<0<b(x_0,t_2)\xi_2 \text{ for any } c<t_1<c+\varepsilon,$$
$$d-\varepsilon<t_2<d \text{ and } |\xi_i-1|<\varepsilon\,(i=1,2).$$

Thus to prove Theorem 1, it suffices to find an open subset $U\subset\Omega_\varepsilon\subset\Omega$ so that the closure of U is compact in Ω_ε and to find functions f_τ, v_τ, depending on a real parameter τ, belonging to $C_0^\infty(U)$ such that

(3) $\lim_{\tau\to\infty}|f_\tau\bar{v}_\tau dxdt|=\infty$

(4) $\lim_{\tau\to\infty}\sum_{|\alpha|\leqq k}\sup|\partial^\alpha f_\tau|<\infty$

(5) $\lim_{\tau\to\infty}\sum_{\beta\leqq N}\sup|\partial^\beta {}^*Pv_\tau|<\infty$

so that the inequality (1) necessary for local solvability is violated.

2. Lemma

According to (5), the function v_τ should be the approximate solution of ${}^*Pv=0$. We note that ${}^*P=-P_0+g$, where $P_0=\frac{\partial}{\partial t}-ib(x,t)\frac{\partial}{\partial x}$ and $g=i\frac{\partial b}{\partial x}+\bar{f}$. As a first step we shall find an approximate solution of the characteristic equation $P_0\phi=0$.

Assume that

$$\begin{aligned}(6)\qquad \phi(x,t)=&\phi_0(t)-i\lambda(t)(x-L(t))+\alpha(t)(x-L(t))^2\\&+\sum_{j=3}^{q}\gamma_j(t)(x-L(t))^j+O(|x-L(t)|^{q+1})\end{aligned}$$

be the approximate solution of $P_0\phi=0$ in the neighborhood of $(x_0,c_0)\in\Omega$

where $c_0 \in (c, d)$ will be determined later. In the expression of $\phi(x,t)$ we assume that $\lambda(t)$ and $L(t)$ are real valued and $\phi_0(t)$, $\alpha(t)$, and $\gamma_j(t)$ $(j=3, 4, \ldots, q)$ are complex valued functions defined in the neighborhood of $t=c_0$. Let us expand $b(x,t)$ such as

$$b(x,t)=\sum_{j=0}^{q-1} \frac{\partial^j b}{\partial x^j}(L(t),t)(x-L(t))^j+O(|x-L(t)|^q).$$

We shall set $O(|x-L(t)|^k)=O_k$, $\frac{\partial^j b}{\partial x^j}(L(t),\ t)=\beta_j$. We solve $P_0\phi=0$ modulo O_q as follows. From $P_0\phi=0$, we get

$$\begin{aligned}&\phi_0'-i\lambda'(x-L)-i\lambda(-L')+\alpha'(x-L)^2+2\alpha(x-L)(-L')+3\gamma_3(x-L)^2\\&\quad(-L')+\cdots+q\gamma_q(x-L)^{q-1}(-L')+\gamma_3'(x-L)^3+\cdots+\gamma_q(x-L)^q+O_q\\&\quad-i[\beta_0+i\beta_1(x-L)+\beta_2(x-L)^2+\cdots+\beta_{q-1}(x-L)^{q-1}+O_q]\times\\&\quad[-i\lambda+2\alpha(x-L)+3\gamma_3(x-L)^2+\cdots+q\gamma_q(x-L)^{q-1}+O_q]=0\end{aligned}$$

(Here $'$ denotes derivative with respect to t variable)
Setting coefficients of $(x-L)^j$ $(j=0, 1, 2, \cdots, q-1)$ to be zero, we have

$$\begin{aligned}(7)\quad&\phi_0'+i\lambda L'-\lambda\beta_0=0\\&i\lambda'+2\alpha L'+i\beta_0(2\alpha)+\beta_1\lambda=0\\&\alpha'-3\gamma_3L'-3i\beta_0\gamma_3-\beta_2\lambda-2i\alpha\beta_1=0\\&\gamma_3'-4\gamma_4L'-i[4\gamma_4\beta_0-i\lambda\beta_3+3\beta_1\gamma_3+2\alpha\beta_2]=0\\&\cdots\cdots\cdots\cdots\end{aligned}$$

Or, equivalently,

$$\begin{aligned}(8)\quad&\mathrm{Re}\ \phi_0'-\lambda\beta_0=0\\&\mathrm{Im}\ \phi_0'+\lambda L'=0\end{aligned}$$

$$\begin{aligned}(9)\quad&\lambda'+2(\mathrm{Im}\ \alpha)L'+2\beta_0(\mathrm{Re}\ \alpha)=0\\&2(\mathrm{Re}\ \alpha)L'-2\beta_0\ (\mathrm{Im}\ \alpha)+\beta_1\lambda=0\\&\mathrm{Re}\ \alpha'-3(\mathrm{Re}\ \gamma_3)L'+3\beta_0\ (\mathrm{Im}\ \gamma_3)-\beta_2\lambda+2\beta_1(\mathrm{Im}\ \alpha)=0\\&\mathrm{Im}\ \alpha'-3(\mathrm{Im}\ \gamma_3)L'-3\beta_0(\mathrm{Re}\ \gamma_3)-2(\mathrm{Im}\ \alpha)\beta_1=0\\&\mathrm{Re}\ \gamma_3'-4(\mathrm{Re}\ \gamma_4)L'+\cdots=0\\&\mathrm{Im}\ \gamma_3'-4(\mathrm{Im}\ \gamma_4)L'+\cdots=0\\&\cdots\cdots\cdots\cdots\\&\mathrm{Re}\ \gamma'_{q-1}-q(\mathrm{Re}\ \gamma_q)L'+\cdots=0\\&\mathrm{Im}\ \gamma'_{q-1}-q(\mathrm{Im}\ \gamma_q)L'+\cdots=0\end{aligned}$$

Now the latter system (9) of $2(q-1)$ linear partial differential equations of λ, L, Re α, Im α, Re γ_j, Im γ_j $(j=3, 4, \cdots, q-1)$ has the leading term, represented by matrix, as

$$\begin{pmatrix} 1 & 2\,\mathrm{Im}\,\alpha & 0 \cdots\cdots\cdots\cdots 0 \\ 0 & 2\,\mathrm{Re}\,\alpha & 0 \cdots\cdots\cdots\cdots 0 \\ 0 & -3\,\mathrm{Re}\,\gamma_3 & 1\ 0 \cdots\cdots\cdots\cdots 0 \\ 0 & -3\,\mathrm{Im}\,\gamma_3 & 0\ 1\ 0 \cdots\cdots\cdots 0 \\ 0 & -4\,\mathrm{Re}\,\gamma_4 & 0\ 0\ 1\ 0 \cdots\cdots 0 \\ 0 & -4\,\mathrm{Im}\,\gamma_4 & 0\ 0\ 0\ 1\ 0 \cdots 0 \\ \cdots & \cdots & \\ 0 & -q\,\mathrm{Re}\,\gamma_q & 0 \cdots\cdots\cdots\cdots 1\ 0 \\ 0 & -q\,\mathrm{Im}\,\gamma_q & 0 \cdots\cdots\cdots\cdots 0\ 1 \end{pmatrix} \begin{pmatrix} \lambda' \\ L' \\ \mathrm{Re}\,\alpha' \\ \mathrm{Im}\,\alpha' \\ \mathrm{Re}\,\gamma_3' \\ \mathrm{Im}\,\gamma_3' \\ \cdots \\ \mathrm{Re}\,\gamma'_{q-1} \\ \mathrm{Im}\,\gamma'_{q-1} \end{pmatrix}$$

Therefore if Re $\alpha\neq 0$, the matrix is nonsingular and since $b(x,t)$ is a C^∞ function, this determined system of linear partial differential equations always has a solution locally.

New let c_0 be the smallest s so that $t\geqq s$ implies $b(x_0,t)\geqq 0$. Let us set $\gamma_q\equiv 0$ in the neighborhood of c_0, chosen later, and solve this system of differential equations in the region

$$|1-\lambda(t)|<\varepsilon,\quad |L-x_0|<\varepsilon,\quad \mathrm{Re}\ \alpha(t)>1/2$$

on $\{t\,|\,c<\tilde{c}<t<\tilde{d}<d\}$, a suitable neighborhood of c_0, with the initial conditions

(10) $\lambda(c_0)=\eta,\ \ L(c_0)=y,\ \ \alpha(c_0)=\zeta_2$
$\gamma_j(c_0)=\zeta_j \qquad (j=3,4,\cdots,q-1),$

where

$$|y-x_0|<\delta',\quad |\eta-1|<\delta',\quad |\zeta_j-\zeta_j^0|<\delta'.$$

Here Re $\zeta_2^0=1$ and Im ζ_2^0, ζ_j^0 $(j=3,\cdots,q-1)$ and δ' is chosen so that the above initial value problem has a solution in the region indicated above. Since the variation of the solutions of (9) can be extimated *a priori* on

$$|\lambda(t)-1|<\varepsilon,\quad |L-x_0|<\varepsilon,\quad c<t<d$$

this choice is possible so that Re $\alpha>1/2$ and the solutions exist on $\tilde{c}<t<\tilde{d}$ for any triple (y,η,ζ) satisfying the given conditions where $\zeta=(\zeta_2,\zeta_3,\cdots,\zeta_{q-1})$. Thus, in particular, δ' should be sufficiently small.

From now on we fix ζ and for each (y,η) we set the solution curve to be

(11) $\qquad x=L(t,y,\eta),\quad \xi=\lambda(t,y,\eta),\quad (\tilde{c}<t<\tilde{d}).$

We claim that at least for one pair (y,η), suitably selected, the function

$$g(t,y,\eta)=-b(L(t,y,\eta),t)\lambda(t,y,\eta)$$

changes sign along the curve (11). That is, it is possible to choose (y,η) and c', d, $(\tilde{c}<c'<d'<d)$ so that

$$b(L(c'),\ c')\lambda(c')<0<b(L(d'),\ d')\lambda(d').$$

In fact, suppose not. Then $b(x,t)\xi$ cannot change sign along the solution curve $x=L(t,y,\eta)$, $\xi=\lambda(t,y,\eta)$ for any (y,η). We may state this as the assumption $g(s,y,\eta)<0$ implies that

$g(t, y, \eta) \leqq 0$ for $t \geqq s$. Now the curve $x=x_0$, $\xi=1$ in the (x, t, ξ) coordinates takes the form, in (t, y, η) coordinates, as

$$L(t, y, \eta)=x_0, \quad \lambda(t, y, \eta)=1.$$

Thus this curve in (t, y, η) coordinates is the integral curve of the differential equation

$$L'+L_y y'+L_\eta \eta'=0$$
$$\lambda'+\lambda_y y'+\lambda_\eta \eta'=0,$$

or, equivalently

$$\begin{bmatrix} y' \\ \eta' \end{bmatrix} = -\begin{bmatrix} L_y L_\eta \\ \lambda_y\ \lambda_\eta \end{bmatrix}^{-1} \begin{bmatrix} L' \\ \lambda' \end{bmatrix}$$

The 2×2 matrix appearing in the expression is nonsingular since the solution curve exists. Let

$$Y=-\begin{bmatrix} L_y L_\eta \\ \lambda_y \lambda_\eta \end{bmatrix}^{-1} \begin{bmatrix} L' \\ \lambda' \end{bmatrix}$$

We shall prove that

(a) for any point (t, y, η) where $g=0$, we have $Y\cdot\begin{bmatrix} g_y \\ g_\eta \end{bmatrix} \leqq 0$, and

(b) for any point (t, y, η) where $g=0$, $g_y=0$ and $g_\eta=0$, we have $Y=0$.

Proof of (a).

Since $\lambda(t, y, \eta)>0$, $g=0$, i.e.,
$b(L(t, y, \eta),\ t)\lambda(t, y, \eta)=0$ at (t, y, η) implies that $b(L(t, y, \eta),\ t)=0$.
Since $b(L(t, y, \eta)=b_0(L(t, y, \eta),\ t)=\beta_0$, $\beta_0=0$ at (t, y, η). Now, we have

$$Y=-\begin{vmatrix} L_y L_\eta \\ \lambda_y \lambda_\eta \end{vmatrix}^{-1} \begin{bmatrix} \lambda_\eta L'-L_\eta\lambda' \\ -\lambda_y L'+L_y\lambda' \end{bmatrix}$$

Let $D=\begin{vmatrix} L_y L_\eta \\ \lambda_y \lambda_\eta \end{vmatrix}$. Then

$$\begin{aligned} Y\cdot\begin{bmatrix} g_y \\ g_\eta \end{bmatrix} &= -D^{-1}\begin{bmatrix} \lambda_\eta L'-L_\eta\lambda' \\ -\lambda_y L'+L_y\lambda' \end{bmatrix}\cdot\begin{bmatrix} g_y \\ g_\eta \end{bmatrix} \\ &= -D^{-1}\left[(\lambda_\eta L'-L_\eta\lambda')g_y+(-\lambda_y L'+L_y\lambda')g_\eta\right] \\ &= -D^{-1}\left[(\lambda_\eta L'-L_\eta\lambda')\ (b_x L_y+b\lambda_y)+(-\lambda_y L'+L_y\lambda')(b_x L_\eta+b\lambda_\eta)\right], \end{aligned}$$

since $-g_y=b_x L_y+b\lambda_y$ and $-g_\eta=b_x L_\eta+b\lambda_\eta$. Now since $b(L(t, y, \eta), t)=0$ at (t, y, η), we have

$$\begin{aligned} Y\cdot\begin{bmatrix} g_y \\ g_\eta \end{bmatrix} &= D^{-1}b_x L_y(\lambda_\eta L'-L_\eta\lambda')+L_\eta(-\lambda_y L'+L_y\lambda') \\ &= D^{-1}b_x(L_y\lambda_\eta L'-L_\eta\lambda_y L')=D^{-1}b_x L' D=b_x L' \\ &= -\left(\frac{\beta_1\lambda}{2(\mathrm{Re}\ \alpha)}\right)bx=-\frac{b_x^2\lambda}{2(\mathrm{Re}\ \alpha)}. \end{aligned}$$

Here we used the fact that

$$L'=-\frac{\beta_1\lambda}{2(\mathrm{Re}\ \alpha)}=-\frac{b_x\lambda}{2(\mathrm{Re}\ \alpha)}$$

which is derived from (9) since $\beta_0=0$ at (t, y, η). As $\lambda>0$ and Re $\alpha>0$, we have

$Y\cdot\begin{bmatrix}g_y\\ g_\eta\end{bmatrix}\leqq 0$ at (t, y, η) where $g(t, y, \eta)=0$.

Proof of (*b*).

From $g=0$ at (t, y, η) we have $\beta_0=b(L(t, y, \eta),\ t)\ \lambda(t, y, \eta)=0$.
Now since $-g_y=b_xL_y+b\lambda_y$ and $-g_\eta=b_xL_\eta+b\lambda_\eta$, if $g_y=g_\eta=0$ at (t, y, η), then we have

$$\beta_1=b_x(L(t, y, \eta),\ t)\ \lambda(t, y, \eta)=0.$$

Thus from (9)

$$\begin{bmatrix}L'\\ \lambda'\end{bmatrix}=\begin{pmatrix}-\dfrac{2\beta_0}{2\mathrm{Re}\ \alpha} & \mathrm{Im}\ \alpha-\dfrac{\beta_1\lambda}{2\mathrm{Re}\ \alpha}\\ -2(\mathrm{Im}\ \alpha)L' & -2\beta_0\ (\mathrm{Re}\ \alpha)\end{pmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

and hence $Y=0$

Now if $g(t, y, \eta)$ does not change sign for any (y, η) on $c'<t<d'$, then by H. Brezis lemma (*cf.* [1]) g does not change sign along the curve

$$L(t, y, \eta)=x_0,\quad \xi(t, y, \eta)=1$$

in (t, y, η) coordinates: which is equivalent to say that $b(x, t)\xi$ does not change sign along the curve $x=x_0$, $\xi=1$ in (x, t, ξ) coordinaties. Since the opposite was assumed in the condition $(\tilde{P})$, it is a contradiction.

Thus there exists a $0<\delta'<\varepsilon$ and real numbers c', d' $(\tilde{c}<c'<d'<\tilde{d})$ so that when

$$|y-y_0|<\delta',\quad |\eta-1|<\delta' \text{ and } |\zeta-\zeta^0|<\delta',$$

the initial value problem (9)-(10) has a solution on $[c',\ d']$ satisfying

$$|L-x_0|<\varepsilon,\quad |\lambda-1|<\varepsilon \text{ and Re } \alpha>1/2 \text{ on } [c', d'].$$

Moreover these numbers can be chosen so that for some (y, η, ζ)

$$b(L(c'),\ c')\lambda(c')<0<b(L(d'),\ d')\lambda(d').$$

From now on for the initial value (10) we take (y, η, ζ) chosen as in the above and shall set the solution curve of (9)-(10) as

(11) $$x=L(t),\quad \lambda=\lambda(t).$$

Now we make a very important remark. From (8) we have Re $\phi_0'=\lambda\beta_0$. Thus Re $\phi_0=\int_{t_0}^{t}\lambda\beta_0\,dt$, where t_0 will be determined later. But

$$\beta_0(c')\lambda(c')=b(L(c', y, \eta, \zeta),\ c')\ \lambda(c', y, \eta, \zeta))<0$$

and

$$\beta_0(d')\lambda(d')=b(L(d', y, \eta, \zeta),\ d')\ \lambda(d', y, \eta, \zeta))>0.$$

This shows that the function $\int_{t_0}^{t}\lambda\beta_0\,\mathrm{dt}$ has the minimum value on $[c', d']$ at $t=r$ where $c'<r<d'$. We set t_0, so far undetermined, as $t_0=r$. By this convention, we have

$$\int_{t_0}^{t} \lambda\beta_0 \, dt \geqq 0 \text{ for any } t \in (c'-\delta, \ d'+\delta)$$

and

$$\int_{t_0}^{t} \lambda\beta_0 \, dt > 0 \text{ at } t=c' \text{ and } t=d'.$$

In choosing d' and c' in the above, we take $(d'-c')$ and δ sufficiently small that

$$\text{Re } \phi_0 \geqq 0 \text{ on } (c'-\delta, \ d'+\delta)$$
$$\text{Re } \phi_0 > 0 \text{ for } t \text{ with } |t-d'|<\delta \text{ or } |t-c'|<\delta.$$

Summing up, we have the following

LEMMA 1. *Suppose that P satisfies the condition $(\tilde{P}_1)$ in Ω. Then for any large integer q, we can find a relatively compact open subset*

$$U=\{(x, t) \mid \ |x-L(t)|<\delta, \ c'-\delta<t<d'+\delta\}$$

of Ω where $0<\delta<\varepsilon$, $c<c'<d'<d$ and a C^∞ function $\phi(x, t)$ defined in U in the form of

$$\phi(x, t)=\phi_0(t)-i\lambda(t)(x-L(t)+\alpha(t)(x-L(t))^2+O(|x-L(t)|^3),$$
$$|x-L(t))| \to 0$$

such that

$$|P_0\phi|=\left|\frac{\partial\phi}{\partial t}-ib(x, t)\frac{\partial\phi}{\partial x}\right| \leqq C|x-L(t)|^q.$$

Here ϕ_0, $\lambda(t)$, $L(t)$ and α satisfy

(i) Re $\phi \geqq 0$ *on* $(c'-\delta, \ d'+\delta)$,

(ii) Re $\phi>0$ *on* $(c'-\delta, \ c'+\delta)$ *and* $(d'-\delta, \ d'+\delta)$,

(iii) $\lambda(t)$ *is a real valued C^∞ function defined on $(c'-\delta, \ d'+\delta)$ such that* $|\lambda(t)-1|<\varepsilon$,

(iv) $\alpha(t)$ *is a complex valued function such that* Re $\alpha(t)>1/2$ *on* $(c'-\delta, \ d'+\delta)$

(v) $L(t)$ *is a real valued C^∞ function defined on $(c'-\delta, \ d'+\delta)$ such that* $|L(t)-x_0|<\varepsilon$.

Once we find the curve $x=L(t)$ in U, shrinking δ if necessary, we can prove the following

LEMMA 2. *Let q be an arbitrary positive integer. There exists a function $h(x, t)$ defined on U, which is a polynomial in $(x-L(t))$ with coefficients depending on t such that*

$$|P_0h-g| \leqq C|x-L(t)|^q \textit{ for some } C>0.$$

Proof. Expand g in a finite Taylor series with remainder

$$g=\sum_{j=0}^{q-1} g_j(t)(x-L(t))^j+G$$

and $G=O(|x-L(t)|^q)=O_q$. Similarly, expand

$$b(x, t)=\sum_{j=0}^{q-1} b_j(t)(x-L(t))^j+B$$

where $B=O_q$ and take for h such a finite expansion in $x-L(t)$, $\sum_{j=0}^{q-1} h_j(t)(x-L(t))^j$ so as to satisfy the differential equation $P_0 h-g=0$ up to order q in $x-L(t)$, i.e., to satisfy the system of ordinary differential equation in the variable t of the coefficients of of the h_j:

$$-L'(j+1)h_{j+1}+\frac{dh_j}{dt}+i\sum_{l=0}^{j}(l+1)b_{j-l}h_{l+1}=g_j \qquad (j=0, \cdots, q-1).$$

Setting $h=0$ when $t_0=r$, we obtain the desired solutions h_j.

3. Proof of the Theorem 1.

Let us first recall that there exists $\delta>0$ such that if $|x-L(t)|<\delta$, then

$$\phi(x, t)=\phi_0-i\lambda(x-L(t))+\alpha(x-L(t))^2+O_3,$$

$$|P_0\phi|=\left|\frac{\partial\phi}{\partial t}-ib\frac{\partial\phi}{\partial x}\right|\leqq \text{const}\cdot|x-L(t)|^q.$$

Thus if δ is sufficiently small, then, setting $a=\text{Re } \alpha$,

$$\text{Re } \phi=\text{Re } \phi_0(t)+a(x-L(t))^2+O_3\geqq \text{Re } \phi_0(t)+\frac{a}{2}(x-L(t))^2.$$

we also recall that

$$\text{Re } \phi_0(t)\geqq\int_{t_0}^{t}\lambda\beta_0\, dt>0$$

on $(c'-\delta, c'+\delta)$ and on $(d'-\delta, d'+\delta)$.
In the sequel we shall assume $(L(t_0), t_0)=(0, 0)$ $\lambda(t_0)=1$, $\phi_0(0)=0$ for the convenience of calculation. Together with the open set

$$U=\{(x, t)\mid |x-L(t)|<\delta,\ c'-\delta<t<d'+\delta\},$$

we define

$$U_0=\left\{(x, t)\in U \mid |x-L(t)|<\frac{\delta}{2},\ c'+\frac{\delta}{2}<t<d'-\frac{\delta}{2}\right\}.$$

Now let us choose a function $\zeta\in C_0^\infty(U)$ such that $\zeta\equiv1$ on U_0. Note that in the support of grad ζ we have Re $\zeta>C_0>0$.

With q of Lemma 1 and 2 chosen as $q=2(4+k+N)$, set $v_\tau=\tau^{3+k}e^{-\tau\phi+h}\zeta$ and choosing $F\in C_0^\infty(R^2)$, set $f_\tau(x, t)=\tau^{-k}F(\tau x, \tau t)$.
For τ sufficiently large, the support of f_τ will lie in U.

We first note that f_τ obviously satisfies (4). Futhermore by a change of variables we find

$$\tau^{-1}\int f_\tau \bar{v}_\tau dx\ dt = \tau^{-1}\int \tau^{-k}F(\tau x,\ \tau t)\bar{v}_\tau\ dx\ dt$$
$$= \tau^{-1}\int \tau^{-k}F(\tau x, \tau t)\ \zeta e^{-\tau\phi+h}\tau^{k+3}dx\ dt$$
$$= \tau^{-1}\int \tau^{-k}F(x,t)\tau^{k+3}e^{-\tau\phi(\tau^{-1}x,\ \tau^{-1}t)}\zeta(\tau^{-1}x, \tau^{-1}t)e^h(\tau^{-1}dx)(\tau^{-1}dt)$$
$$= \int F(x,t)e^{-\tau\phi(\tau^{-1}x,\ \tau^{-1}t)}\zeta(\tau^{-1}x, \tau^{-1}t)e^h\,dx\ dt$$

But

$$\lim_{\tau\to\infty} \tau\phi(\tau^{-1}x, \tau^{-1}t)$$
$$= \lim_{\tau\to\infty} \frac{1}{\tau^{-1}}\phi_0\left(\frac{t}{\tau}\right) - i\lambda\left(\frac{t}{\tau}\right)\left(\frac{x}{\tau} - L\left(\frac{t}{\tau}\right)\right) + a\left(\frac{x}{\tau} - L\left(\frac{t}{\tau}\right)\right)^2$$
$$+ O\left(\left|\frac{x}{\tau} - L\left(\frac{t}{\tau}\right)\right|^3\right)$$
$$= \phi_0'(0)t - ix + iL'(0)t = -ix,$$

since $\phi_0'(0)t + iL'(0)t = 0$, which follows from

$$\phi_0'(0) = -iL'(0) + \beta_0(0) = -iL'(0) + b(L(0), 0) = -iL'(0).$$

Therefore as $\tau\to\infty$,

$$\tau^{-1}\int f_\tau\bar{v}_\tau\,dx\ dt \to \int F(x,t)e^{-ix}\ dx\ dt = F(1,0).$$

Choosing F so that $F(1,0)$ is not zero, we see that

$$\left|\int f_\tau\bar{v}_\tau\ dx\ dt\right| \to \infty \text{ as } \tau\to\infty.$$

Finally we have to prove (5). Since $^*P = -P_0 + q$, we see that

$$^*Pv_\tau = \tau^{3+k}e^{-\tau\phi+h}(-\tau P_0\phi - P_0 h + g)\zeta - P_0\zeta.$$

Now the last term $-\tau^{3+k}e^{-\tau\phi+h}P_0\zeta$ is such that where it does not vanish Re ϕ is bounded from below by a positive number $C_0 > 0$. Hence this term and it's derivatives go to zero as $\tau\to\infty$.

By the Lemma 1 and 2 the other terms are of the form

$$\tau^{3+k}e^{-\tau\phi+h}\tau O(|x-L(t)|^q) + O(|x-L(t)|^q)$$

Since, for $\beta < N$,

$$\tau^{4+k+\beta}|x-L(t)|^q|e^{-\tau\phi+h}|$$
$$= \tau^{4+k+\beta}|x-L(t)|^q e^{\mathrm{Re}h}e^{-\tau\mathrm{Re}\phi}$$
$$\leqq \tau^{4+k+\beta}|x-L(t)|^q e^{\mathrm{Re}h}e^{-1/2a\tau(x-L(t))^2}$$
$$= \tau^{4+k+\beta}e^{\mathrm{Re}h}2^{\frac{q}{2}}\tau^{-\frac{q}{2}}\left(\tau\frac{|x-L(t)|^2}{2}\right)^{\frac{q}{2}} e^{-1/2a\tau(x-L(t))^2}$$
$$\leqq C\tau^{-\frac{q}{2}+4+k+\beta}e^{-1/4a\tau(x-L(t))^2},$$

and since $-\frac{q}{2}+4+k+\beta < 0$, we see that for any $\beta < N$, $|\partial^\beta {}^*Pv_\tau|$ goes to zero as $\tau\to\infty$, thus proving (5).

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