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GENERIC SUBMANIFOLDS OF A QUATERNIONIC PROJECTIVE SPACE WITH PARALLEL MEAN CURVATURE VECTOR

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1. Introduction

Recently M. Okumura [9], Y. Shibuya [14], R. O. Wells [15] extended real hypersurfaces of Kaehlerian manifolds or quaternionic Kaehlerian manifolds into submanifolds with codimension >1 in such a way that any normal space is mapped in the tangent space of the submanifold under the action of the almost complex structure tensor or that of the almost quaternionic structure of the ambient manifold. Such a submanifold is called a generic submanifold (anti-holomorphic or anti-quaternionic submanifold).

In this point of view, many authors (Y. H. Kim [5], M. Kon [16], U-Hang Ki [5], M. Okumura [10], J. S. Pak [5], [12], [13], Y. Shibuya [14], K. Yano [16], etc.) have actively extended the results due to Lawson [7], Maeda [8], Okumura [9], J. S. Pak [11], etc. for real hypersurfaces.

The purpose of the present paper is to study generic submanifolds with parallel mean curvature vector immersed in a quaternionic projective space by the method of Riemannian fibre bundles and the aid of the following theorems proved by K. Yano and M. Kon [16].

THEOREM A. *Let M be a complete minimal submanifold of dimension n immersed in an $(n+p)$-dimensional unit sphere S^{n+p} with parallel second fundamental form. If the square of the length of the second fundamental form is not less than np, then M is a pythagorean product of the form*

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N),\quad r_t=\sqrt{p_t/n}\quad (t=1,\cdots,N),$$

where $p_1,\ldots,p_N\geqq 1$, $p_1+\cdots+p_N=n$, $p=N-1$.

THEOREM B. *Let M be a complete n-dimensional submanifold of S^m with flat normal connection. If the second fundamental form of M is parallel, then M is a small sphere, a great sphere or a pythagorean product of a certain number of spheres. Moreover, if M is of essential codimension $m-n$,*

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then M is a pythagorean product of the form

$$S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N),\quad r_1^2+\cdots+r_N^2=1,\quad N=m-n+1,$$

or a pythagorean product of the form

$$S^{p_1}(r_1)\times\cdots\times S^{p_{N'}}(r_{N'})\subset S^{m-1}(r)\subset S^m,$$

where $r_1^2+\cdots+r_{N'}^2=r^2<1,\ N'=m-n.$

2. Generic submanifolds of a quaternionic Kaehlerian manifold

Let $\tilde{M}$ be a $4m$-dimensional differentiable manifold covered by a system of coordinate neighborhoods $\{\tilde{U}, y^h\}$ (here and in the sequel the indices h, i, j, k, t, s run over the range $\{1, 2, \ldots, 4m\}$) and assume that there exists a 3-dimensional vector bundle V consisting of tensors of type (1, 1) over $\tilde{M}$ satisfying the following conditions (i), (ii) and (iii):

(i) In any neighborhood $\{\tilde{U}, y^h\}$, there is a local base $\{F, G, H\}$ of V such that

(2.1) $$F_h{}^iF_j{}^h=-\delta_j^i,\ G_h{}^iG_j{}^h=-\delta_j^i,\ H_h{}^iH_j{}^h=-\delta_j^i,\ F_h{}^iG_j{}^h=-G_h{}^iF_j{}^h=H_j{}^i,$$
$$G_h{}^iH_j{}^h=-H_h{}^iG_j{}^h=F_j{}^i,\ H_h{}^iF_j{}^h=-F_h{}^iH_j{}^h=G_j{}^i,$$

where $F_j{}^i$, $G_j{}^i$ and $H_j{}^i$ denoting components of F, G and H in $\tilde{U}$, respectively.

(ii) There is a Riemannian metric tensor g_{ji} such that

(2.2) $$F_{ji}=-F_{ij},\quad G_{ji}=-G_{ij},\quad H_{ji}=-H_{ij},$$

where $F_{ji}=g_{hi}F_j{}^h,\ G_{ji}=-g_{hi}G_j{}^h,\ H_{ji}=g_{hi}H_j{}^h.$

(iii) For the Riemannian connection ∇ of $(\tilde{M}, g)$,

(2.3) $$\begin{aligned}\nabla_jF_i{}^h&=\phantom{-r_jF_i{}^h}\ r_jG_i{}^h-q_jH_i{}^h,\\ \nabla_jG_i{}^h&=-r_jF_i{}^h\phantom{-q_jH_i{}^h}\ +p_jH_i{}^h,\\ \nabla_jH_i{}^h&=q_jF_i{}^h-p_jG_i{}^h,\end{aligned}$$

where $p=p_idy^i$, $q=q_idy^i$ and $r=r_idy^i$ are certain local 1-forms defined in $\tilde{U}$. Such a local base $\{F, G, H\}$ is called a canonical base of the bundle V in $\tilde{U}$ and $(\tilde{M}, g, V)$ or M is called a quaternionic Kaehlerian manifold and (g, V) a quaternionic Kaehlerian structure.

In a quaternionic Kaehlerian manifold (M, g, V) we take intersecting coordinate neighborhoods $\tilde{U}$ and $'\tilde{U}$. Let $\{F, G, H\}$ and $\{'F, 'G, 'H\}$ be canonical local bases of V in $\tilde{U}$ and $'\tilde{U}$ respectively. Then it follows that in $\tilde{U}\cap{}'\tilde{U}$

$$\begin{pmatrix}'F\\ 'G\\ 'H\end{pmatrix}=(s_{xy})\begin{pmatrix}F\\ G\\ H\end{pmatrix},\quad (x, y=1, 2, 3)$$

with differentiable function s_{xy}, where the matrix $s=(s_{xy})$ is contained in the special orthogonal group $SO(3)$ as a consequence of (2.1). As is well known, a quaternionic Kaehlerian manifold is orientable.

From now on we consider an n ($<4m$)-dimensional submanifold M in a quaternionic Kaehlerian manifold $\tilde{M}$ of dimension $4m$. Let M be covered by a system of coordinate neighborhoods $\{\tilde{U}, y^h\}$. Then M is covered by a system of coordinate neighborhoods $\{U, x^a\}$ (here and in the sequel the indices, a, b, c, d, e run over the range $1, 2, \ldots, n$), where $U=\tilde{U}\cap M$. Let M be represented by $y^i=y^i(x^a)$ with respect to local coordinates (y^i) in $\tilde{U}$ and (x^a) in U. We put

$$B_a{}^j=\partial_a y^j \quad (\partial_a=\partial/\partial x_a)$$

and denote by $N_x{}^j$ $4m-n$ mutually orthogonal unit vectors normal to M^n (here and in the sequel the indices x, y, z, w run over the range $\{1, 2, \ldots 4m-n\}$). Then the induced metric tensor g_{ba} of M and g_{xy} that of the normal bundle are respectively given by

$$g_{ba}=g_{ji}B_b{}^jB_a{}^i, \quad g_{xy}=g_{ji}N_x{}^jN_y{}^i.$$

If the transforms by $F_j{}^i, G_j{}^i$ and $H_j{}^i$ of any vector normal to M are tangent to M at the same time, the submanifold M is said to be generic in $\tilde{M}$ (see [14]). Since the ranks of $F_j{}^i$, $G_j{}^i$ and $H_j{}^i$ are all $4m$, we have $n\geqq 3p$, $p=4m-n$.

For a generic submanifold M of $\tilde{M}$, we have equations of the form

$$\begin{aligned} &F_h{}^iB_a{}^h=\phi_a{}^bB_b{}^i+\phi_a{}^xN_x{}^i, \quad F_h{}^iN_x{}^h=-\phi_x{}^aB_a{}^i, \\ &G_h{}^iB_a{}^h=\psi_a{}^bB_b{}^i+\psi_a{}^xN_x{}^i, \quad G_h{}^iN_x{}^h=-\psi_x{}^aB_a{}^i, \\ &H_h{}^iB_a{}^h=\theta_a{}^bB_b{}^i+\theta_a{}^xN_x{}^i, \quad H_h{}^iN_x{}^h=-\theta_x{}^aB_a{}^i. \end{aligned} \tag{2.5}$$

Using (2.2), we have from (2.5)

$$\begin{aligned} &\phi_{ab}=-\phi_{ba}, \quad \psi_{ab}=-\psi_{ba}, \quad \theta_{ab}=-\theta_{ba}, \\ &\phi_{ax}=\phi_{xa}, \quad \psi_{ax}=\psi_{xa}, \quad \theta_{ax}=\theta_{xa}, \end{aligned} \tag{2.6}$$

where $\phi_{ab}=\phi_a{}^cg_{cb}$ and $\phi_{ax}=\phi_a{}^yg_{yx}$.

Applying F, G, H to (2.5) and using (2.1) and those equations, it follows that

$$\begin{aligned} &\phi_c{}^b\phi_a{}^c=-\delta_a{}^b+\phi_a{}^x\phi_x{}^b, \quad \psi_c{}^b\psi_a{}^c=-\delta_a{}^b+\psi_a{}^x\psi_x{}^b, \quad \theta_c{}^b\theta_a{}^c=-\delta_a{}^b+\theta_a{}^x\theta_x{}^b, \\ &\psi_c{}^b\phi_a{}^c=-\theta_a{}^b+\phi_a{}^x\psi_x{}^b, \quad \theta_c{}^b\phi_a{}^c=\psi_a{}^b+\phi_a{}^x\theta_x{}^b, \quad \phi_a{}^b\phi_b{}^x=0, \\ &\theta_c{}^b\psi_a{}^c=-\phi_a{}^b+\psi_a{}^x\theta_x{}^b, \quad \phi_c{}^b\psi_a{}^c=\theta_a{}^b+\psi_a{}^x\phi_x{}^b, \quad \psi_a{}^b\psi_b{}^x=0, \\ &\phi_c{}^b\theta_a{}^c=-\psi_a{}^b+\theta_a{}^x\phi_x{}^b, \quad \psi_c{}^b\theta_a{}^c=\phi_a{}^b+\theta_a{}^x\psi_x{}^b, \quad \theta_a{}^b\theta_b{}^x=0, \\ &\phi_a{}^c\psi_c{}^x=-\theta_a{}^x, \quad \phi_a{}^c\theta_c{}^x=\psi_a{}^x, \quad \psi_a{}^c\theta_c{}^x=-\phi_a{}^x, \quad \psi_a{}^c\phi_c{}^x=\theta_a{}^x, \\ &\theta_a{}^c\phi_c{}^x=-\psi_a{}^x, \quad \theta_a{}^c\psi_c{}^x=\phi_a{}^x, \quad \phi_x{}^a\psi_a{}^y=0, \quad \psi_x{}^a\theta_a{}^y=0, \quad \theta_x{}^a\phi_a{}^y=0, \\ &\phi_x{}^a\phi_a{}^y=\delta_x{}^y, \quad \psi_x{}^a\psi_a{}^y=\delta_x{}^y, \quad \theta_x{}^a\theta_a{}^y=\delta_x{}^y. \end{aligned} \tag{2.7}$$

We denote by ∇ the Riemannian connection induced on M from the connection of $\tilde{M}$. Then equations of Gauss and Weingarten are given by

$$\nabla_bB_a{}^i=h_{ba}{}^xN_x{}^i, \quad \nabla_bN_x{}^i=-h_b{}^a{}_xB_a{}^i \tag{2.8}$$

respectively, where $h_{ba}{}^x$ are the components of the second fundamental tensor with respect to the unit normal vector $N_x{}^i$ and $h_b{}^a{}_x=h_{bc}{}^yg^{ca}g_{yx}$.

Applying the operator $\nabla_a = B_a{}^i\nabla_i$ to (2.5) and using (2.3) and (2.8), we can find

$$\begin{aligned}
&(2.9) && \nabla_c\phi_a{}^b = r_c\psi_a{}^b - q_c\theta_a{}^b + h_c{}^b{}_x\phi_a{}^x - h_{ba}{}^x\phi_x{}^b,\\
& && \nabla_c\phi_a{}^x = r_c\psi_a{}^x - q_c\theta_a{}^x - h_{cb}{}^x\phi_a{}^b,\quad h_{cx}{}^e\phi_e{}^y = h_{ce}{}^y\phi_x{}^e,\\
&(2.10) && \nabla_c\psi_a{}^b = -r_c\phi_a{}^b + p_c\theta_a{}^b + h_c{}^b{}_x\psi_a{}^x - h_{ca}{}^x\psi_x{}^b,\\
& && \nabla_c\psi_a{}^x = -r_c\phi_a{}^x + p_c\theta_a{}^x - h_{cb}{}^x\psi_a{}^b,\quad h_{cx}{}^e\psi_e{}^y = h_{ce}{}^y\psi_x{}^e,\\
&(2.11) && \nabla_c\theta_a{}^b = q_c\phi_a{}^b - p_c\psi_a{}^b + h_c{}^b{}_x\theta_a{}^x - h_{ca}{}^x\theta_x{}^b,\\
& && \nabla_c\theta_a{}^x = q_c\phi_a{}^x - p_c\psi_a{}^x - h_{cb}{}^x\theta_a{}^b,\quad h_c{}^e{}_x\theta_e{}^y = h_{ce}{}^y\theta_x{}^e,
\end{aligned}$$

where

$$(2.12)\qquad p_c = B_c{}^ip_i,\quad q_c = B_c{}^iq_i,\quad r_c = B_c{}^ir_i.$$

We now consider intersections of coordinate neighborhoods $U = \tilde U \cap M$ and $'U = '\tilde U \cap M$. Then, taking account of (2.4) and of (2.5) established in $U \cap 'U$, we can prove that

$$(2.13)\qquad \begin{pmatrix} '\phi_{ab} \\ '\psi_{ab} \\ '\theta_{ab} \end{pmatrix} = (s_{xy}) \begin{pmatrix} \phi_{ab} \\ \psi_{ab} \\ \theta_{ab} \end{pmatrix},\qquad \begin{pmatrix} '\phi_{ax} \\ '\psi_{ax} \\ '\theta_{ax} \end{pmatrix} = (s_{xy}) \begin{pmatrix} \phi_{ax} \\ \psi_{ax} \\ \theta_{ax} \end{pmatrix}$$

hold in $U \cap 'U$, where the restriction of function s_{xy} defined in $\tilde U \cap '\tilde U$ to $U \cap 'U$ is denoted by the same letter s_{xy}. If we now put

$$\begin{aligned}
&\mathring\nabla_c\phi_a{}^b = \nabla_c\phi_a{}^b - r_c\psi_a{}^b + q_c\theta_a{}^b,\quad \mathring\nabla_c\phi_a{}^x = \nabla_c\phi_a{}^x - r_c\psi_a{}^x + q_c\theta_a{}^x,\\
&\mathring\nabla_c\psi_a{}^b = \nabla_c\psi_a{}^b + r_c\phi_a{}^b - p_c\theta_a{}^b,\quad \mathring\nabla_c\psi_a{}^x = \nabla_c\psi_a{}^x + r_c\phi_a{}^x - p_c\theta_a{}^x,\\
&\mathring\nabla_c\theta_a{}^b = \nabla_c\theta_a{}^b - q_c\phi_a{}^b + p_c\psi_a{}^b,\quad \mathring\nabla_c\theta_a{}^x = \nabla_c\theta_a{}^x - q_c\phi_a{}^x + p_c\psi_a{}^x,
\end{aligned}$$

then from (2.13) we have

$$(2.14)\qquad \begin{pmatrix} \mathring\nabla'\phi_{ab} \\ \mathring\nabla'\psi_{ab} \\ \mathring\nabla'\theta_{ab} \end{pmatrix} = (s_{xy}) \begin{pmatrix} \mathring\nabla\phi_{ab} \\ \mathring\nabla\psi_{ab} \\ \mathring\nabla\theta_{ab} \end{pmatrix},\qquad \begin{pmatrix} \mathring\nabla'\phi_{ax} \\ \mathring\nabla'\psi_{ax} \\ \mathring\nabla'\theta_{ax} \end{pmatrix} = (s_{xy}) \begin{pmatrix} \mathring\nabla\phi_{ax} \\ \mathring\nabla\psi_{ax} \\ \mathring\nabla\theta_{ax} \end{pmatrix}$$

in $U \cap 'U$. On the other hand, (2.9), (2.10) and (2.11) give respectively

$$\begin{aligned}
&(2.15) && \mathring\nabla_c\phi_a{}^b = h_c{}^b{}_x\phi_a{}^x - h_{ca}{}^x\phi_x{}^b,\quad \mathring\nabla_c\phi_a{}^x = -h_{ae}{}^x\phi_a{}^e,\\
&(2.16) && \mathring\nabla_c\psi_a{}^b = h_c{}^b{}_x\psi_a{}^x - h_{ca}{}^x\psi_x{}^b,\quad \mathring\nabla_c\psi_a{}^x = -h_{ce}{}^x\psi_a{}^e,\\
&(2.17) && \mathring\nabla_c\theta_a{}^b = h_c{}^b{}_x\theta_a{}^x - h_{ca}{}^x\theta_x{}^b,\quad \mathring\nabla_c\theta_a{}^x = -h_{ce}{}^x\theta_a{}^e.
\end{aligned}$$

We now assume that the ambient manifold $\tilde M$ is a quaternionic Kaehlerian manifold with constant Q-sectional curvature c. Then the components $K_{kji}{}^h$ of the curvature tensor of $\tilde M$ are of the form

$$\begin{aligned}
K_{kji}{}^h = \frac{c}{4}(&\delta_k^hg_{ji} - \delta_j^hg_{ki} + F_k{}^hF_{ji} - F_j{}^hF_{ki} - 2F_{kj}F_i{}^h + G_k{}^hG_{ji} - G_j{}^hG_{ki}\\
&-2G_{kj}G_i{}^h + H_k{}^hH_{ji} - H_j{}^hH_{ki} - 2H_{kj}H_i{}^h),
\end{aligned}$$

where c is necessarily a constant, provided $m \geqq 2$ (see [2]). Hence the structure equations of Gauss, Codazzi and Ricci are given respectively by

$$K_{dcb}{}^{a}=\frac{c}{4}(\delta_{d}{}^{g}g_{cb}-\delta_{c}{}^{a}g_{db}+\phi_{d}{}^{a}\phi_{cb}-\phi_{c}{}^{a}\phi_{db}-2\phi_{dc}\phi_{b}{}^{a}+\psi_{d}{}^{a}\psi_{cb}$$
$$-\psi_{c}{}^{a}\psi_{db}-2\psi_{dc}\psi_{b}{}^{a}+\theta_{d}{}^{a}\theta_{cb}-\theta_{c}{}^{a}\theta_{db}-2\theta_{dc}\theta_{b}{}^{a})+h_{d}{}^{a}{}_{x}h_{db}{}^{x}-h_{c}{}^{a}{}_{x}h_{db}{}^{x}, \tag{2.18}$$

$$\nabla_{c}h_{ba}{}^{x}-\nabla_{b}h_{ca}{}^{x}=\frac{c}{4}(\phi_{c}{}^{x}\phi_{ba}-\phi_{b}{}^{x}\phi_{ca}-2\phi_{cb}\phi_{a}{}^{x}+\psi_{c}{}^{x}\psi_{ba}-\psi_{b}{}^{x}\psi_{ca}$$
$$-2\psi_{cb}\psi_{a}{}^{x}+\theta_{c}{}^{x}\theta_{ba}-\theta_{b}{}^{x}\theta_{ca}-2\theta_{cb}\theta_{a}{}^{x}), \tag{2.19}$$

$$K_{cby}{}^{x}=\frac{c}{4}(\phi_{c}{}^{x}\phi_{by}-\phi_{b}{}^{x}\phi_{cy}+\psi_{c}{}^{x}\psi_{by}-\psi_{b}{}^{x}\psi_{cy}+\theta_{c}{}^{x}\theta_{by}-\theta_{b}{}^{x}\theta_{cy})$$
$$+h_{ca}{}^{x}h_{b}{}^{a}{}_{y}-h_{ba}{}^{x}h_{c}{}^{a}{}_{y}, \tag{2.20}$$

where $K_{dcb}{}^{a}$ and $K_{cby}{}^{x}$ denote components of the curvature tensors determined by the induced metric g_{cb} and g_{xy} in M and in the normal bundle of M respectively.

3. Submersion $\tilde{\pi}: S^{4m+3}\to QP^{m}$ and immersion $i: M\to QP^{m}$

We denote by $\tilde{\pi}$ the natural projection of S^{4m+3} (1) onto a quaternionic projective space QP^{m} which is defined by the Hopf-fibration $S^{3}\to S^{4m+3}\to QP^{m}$ As is well known, since S^{4m+3} admits a Sasakian 3-structure ξ, η, ζ and any fibre $\tilde{\pi}^{-1}(P)$, $P\in QP^{m}$, is a maximal integral manifold of the distribution spanned by ξ, η and ζ, the base space QP^{m} of a fibred Riemannian space with Sasakian 3-structure admits the induced a quaternionic Kaehlerian structure, and moreover is of constant Q-sectional curvature 4 (see [1], [2]).

We consider a Riemannian submersion $\pi:\bar{M}\to M$ with totally geodesic fibres such that the following diagram is commutative:

$$\begin{array}{ccc} \bar{M} & \xrightarrow{\ \tilde{i}\ } & S^{4m+3} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ M & \xrightarrow[\ i\ ]{} & QP^{m} \end{array}$$

Here $\bar{M}=\tilde{\pi}^{-1}(M)$ is a submanifold of real codimension p in S^{4m+3}, M that of QP^{m}, and $i:\bar{M}\to S^{4m+3}$ and $i: M\to QP^{m}$ are certain isometric immersions.

We now take coordinate neighborhoods $\{\bar{U}, x^{\alpha}\}$ of $\bar{M}$ such that $\pi(\bar{U})=U$ are coordinate neighborhoods of M with local coordinate (x^{a}). Then the projection $\pi:\bar{M}\to M$ may be expressed by

$$x^{a}=x^{a}(x^{\alpha}) \tag{3.1}$$

and the Jacobian $(\partial x^{a}/\partial x^{\alpha})$ of the function $x^{a}(x^{\alpha})$ has the maximum rank $n=4m-p$ We take a fibre Γ such that $\Gamma\cap\bar{U}\neq 0$. Then we can take local coordinates (z^{s}) in $\Gamma\cap\bar{U}$ in such a way that (x^{a}, z^{s}) is a system of local coordinates in $\bar{U}$, (x^{a}) being coordinates of $\pi(\Gamma)$ in U. Differentiating

(3. 1) with respect to x^α, we put

$$E_\alpha{}^a = \partial_\alpha x^a \quad (\partial_\alpha = \partial/\partial x^\alpha)$$

and denote by E^a local covector fields with components $E_\alpha{}^a$ in $\bar{U}$. On the other hand $C_s{}^\alpha = \partial/\partial z^s$ form a natural frame tangent to each fibre Γ in $\Gamma \cap \bar{U}$. Denoting by $C^\alpha{}_s$ components of C_s in $\bar{U}$, we put $C_\alpha{}^s = g_{\alpha\beta} g^{st} C^\beta{}_t$, where $g_{\alpha\beta}$ are components of the induced metric of $\bar{M}$ from that of S^{4m+3} in $\bar{U}$, $g_{st} = g_{\alpha\beta} C^\alpha{}_s C^\beta{}_t$ and $(g^{ts}) = (g_{st})^{-1}$. We now denote by C^s local covector fields with components $C_\alpha{}^s$ in $\bar{U}$. We next define $E^\alpha{}_a$ by

$$(E^\alpha{}_a, C^\alpha{}_s) = (E^a{}_\alpha, C_\alpha{}^s)^{-1}$$

and denote by E_a local vector fields with components $E^\alpha{}_a$ in $\bar{U}$. Then $\{E_b, C_s\}$ is a local frame in $\bar{U}$ and $\{E^b, C^s\}$ the coframe dual to $\{E_b, C_s\}$ in $\bar{U}$.

We now take coordinate neighborhoods $\{\tilde{U}, y^\kappa\}$ of S^{4m+3} such that $\tilde{\pi}(\tilde{U}) = \tilde{U}$ are coordinate neighborhoods of QP^m with local coordinate (y^i). Then we can also define a local frame $\{E_j, C_s\}$ and the coframe $\{E^j, C^s\}$ dual to $\{E_j, C_s\}$ in $\tilde{U}$ similarly (see Ishihara [1], [2], [3], [4] and Konishi [3], [4]).

We denote by $\{E^\kappa{}_j, C^\kappa{}_s\}$ and $\{E_\kappa{}^j, C_\kappa{}^s\}$ components of $\{E_j, C_s\}$ and $\{E^j, C^s\}$ respectively in $\tilde{U}$. Let the isometric immersion $\bar{i}$ and i be locally expressed by $y^\kappa = y^\kappa(x^\alpha)$ and $y^j = y^j(x^a)$ respectively. Then the commutativity of the diagram implies

$$y^i(x^a(x^\alpha)) = y^i(y^\kappa(x^\alpha)),$$

and consequently

$$B_a{}^i E_\alpha{}^a = E_\kappa{}^i B_\alpha{}^\kappa, \tag{3.2}$$

where $B_a{}^i = \partial_a y^i$ and $B_\alpha{}^\kappa = \partial_\alpha y^\kappa$.

For an arbitrary point $P_\varepsilon \in M$ we choose a unit normal vector field $N_x{}^i$ to M defined in a neighborhood U of P in such a way that $\{B_a{}^i, N_x{}^i\}$ span the tangent space of QP^m at $i(\bar{P})$. Let $\bar{P}$ be an arbitrary point of the fibre Γ over P, then the lifts $N_x{}^\kappa = N_x{}^i E^\kappa{}_i$ of $N_x{}^i$ is a unit normal vector to $\bar{M}$ defined in the tubular neighborhood over U because of (3. 2).

Let's denote by ξ^κ, η^κ and η^κ components of ξ, η and ζ of the induced Sasakian 3-structure $\{\xi, \eta, \zeta\}$ in S^{4m+3} respectively. Since any fibre $\tilde{\pi}^1(P)$, $P \in QP^m$, is a maximal integral manifold of the distribution spanned by ξ, η and ζ, ξ^κ, η^κ and ζ^κ can be represented by

$$\xi^\kappa = \xi^\alpha B_\alpha{}^\kappa, \quad \eta^\kappa = \eta^\alpha B_\alpha{}^\kappa, \quad \zeta^\kappa = \zeta^\alpha B_\alpha{}^\kappa, \tag{3.3}$$

where ξ^α, η^α and ζ^α are unit vector fields in $\bar{M}$ which are vertical and span the tangent space to the fibre Γ at each point of M because of (3. 2). We now put in $\tilde{U}$

$$\xi = a^s C_s, \quad \eta = b^s C_s, \quad \zeta = c^s C_s,$$
$$a_s = a^t g_{ts}, \quad b_s = b^t g_{ts}, \quad c_s = c^t g_{ts},$$

where $g_{ts}=g_{\lambda\mu}C_t{}^\lambda C^\mu{}_s$ and $g_{\lambda\mu}$ components of the induced metric in $S^{4m+3}(\subset Q^{m+1})$. Then it follows that

(3.4) $$C_s=a_s{}^\xi+b_s{}^\eta+c_s{}^\zeta,$$

(3.5) $$a_s a^t+b_s b^t+c_s c^t=\delta_s{}^t.$$

Transvecting (3.2) with $E^\mu{}_j$ and substituting (3.4) imply

$$(E^\mu{}_j B_a{}^j)E_\alpha{}^a=B_\alpha{}^\mu-(a^s\xi_\alpha+b^s\eta_\alpha+c^s\zeta_\alpha)C^\mu{}_s,$$

where $\xi_\alpha=\xi^\beta g_{\beta\alpha}$, $\eta_\alpha=\eta^\beta g_{\beta\alpha}$ and $\zeta_\alpha=\zeta^\beta g_{\alpha\beta}$. Thus, transvecting the above equation with $E^\alpha{}_b$ and using the fact that ξ_α, η_α and ζ_α being vertical, we have

(3.6) $$E^\mu{}_j B_b{}^j=B_\alpha{}^\mu E^\alpha{}_b.$$

Hence the vertical vectors C_s can be written as

(3.7) $$C_s=a_s\xi+b_s\eta+c_s\zeta$$

in such a way that the fuctions a_s, b_s and c_s satisfy (3.5), where a_s, b_s and c_s are respectively the restrictions of a_s, b_s and c_s appearing in (3.4) and in the sequel these restrictions will be denoted by the corresponding letters, respectively.

Denoting by $\{{}_\mu{}^\kappa{}_\nu\}$, $\{{}_j{}^i{}_h\}$, $\{{}_\beta{}^\alpha{}_\gamma\}$ and $\{{}_b{}^a{}_c\}$ the Christoffel symbols formed with respect to the Riemannian metrics $g_{\lambda\mu}, g_{ji}, g_{\beta\alpha}$ and g_{ba} respectively, we have

$$D_\mu E_\lambda{}^i=\partial_\mu E_\lambda{}^i-\{{}_\mu{}^\kappa{}_\lambda\}E_\kappa{}^i+\{{}_j{}^i{}_h\}E_\mu{}^j E_\lambda{}^h,$$
$$D_\mu E^\lambda{}_i=\partial_\mu E^\lambda{}_i+\{{}_\mu{}^\lambda{}_\kappa\}E^\kappa{}_i-\{{}_i{}^h{}_j\}E_\mu{}^j E^\lambda{}_h$$

and

$$\nabla_\beta E_\alpha{}^a=\partial_\beta E_\alpha{}^a-\{{}_\beta{}^\gamma{}_\alpha\}E_\gamma{}^a+\{{}_b{}^a{}_c\}E_\beta{}^b E_\alpha{}^c,$$
$$\nabla_\beta E^\alpha{}_a=\partial_\beta E^\alpha{}_a+\{{}_\beta{}^\alpha{}_\gamma\}E^\gamma{}_a-\{{}_b{}^c{}_a\}E_\beta{}^b E^\alpha{}_c.$$

Since the metrics $g_{\lambda\mu}$ and $g_{\alpha\beta}$ are invariant with respect to the submersions $\tilde{\pi}$ and π respectively the van der Waerden-Bortolotti covariant derivatives of $E_\lambda{}^i, E^\lambda{}_i$ and $E_\alpha{}^a, E^\alpha{}_a$ are given by

(3.8) $$D_\mu E_\lambda{}^i=A_j{}^i{}_s(E_\mu{}^j C_\lambda{}^s+C_\mu{}^s E_\lambda{}^j),\quad D_\mu E^\lambda{}_i=A_{ji}{}^s E_\mu{}^j C^\lambda{}_s-A_i{}^j{}_s C_\mu{}^s E^\lambda{}_j,$$

(3.9) $$\nabla_\beta E_\alpha{}^a=A_b{}^a{}_s(E_\beta{}^b C_\alpha{}^s+C_\beta{}^s E_\alpha{}^b),\quad \nabla_\beta E^\alpha{}_a=A_{ba}{}^s E_\beta{}^b C^\alpha{}_s-A_a{}^b{}_s C_\beta{}^s E^\alpha{}_b$$

respectively, where $A_j{}^i{}_s=g^{ih}g_{st}A_{jh}{}^t$, $A_b{}^a{}_s=g^{ac}g_{st}A_{bc}{}^t$, $A_{ji}{}^s$ being $A_{ba}{}^s$ are the structure tensors induced from the submersion $\tilde{\pi}$ and π, respectively (see Ishihara and Konishi [4]).

On the other hand, the equations of Gauss and Weingarten for the immersion $i: M\to S^{4m+3}$ are given by

(3.10) $$\nabla_\beta B_\alpha{}^\kappa=h_{\beta\alpha}{}^x N_x{}^\kappa,\quad \nabla_\beta N_x{}^\kappa=-h_\beta{}^\alpha{}_x B_\alpha{}^\kappa,$$

and those for the immersion $i: M\to QP^m$ by

(3.11) $$\nabla_b B_a{}^i=h_{ba}{}^x N_x{}^i,\quad \nabla_b N_x{}^i=-h_b{}^a{}_x B_a{}^i,$$

where $h_\beta{}^\alpha{}_x=h_{\beta\gamma}{}^y g^{\gamma\alpha}g_{xy}$, $h_{\beta\alpha}{}^x$ being $h_{ba}{}^x$ are the second fundamental tensors of $\overline{M}$ and M with respect to the unit normal vectors $N_x{}^k$ and $N_x{}^j$ respectively. Moreover in this case (3.2) and (3.6) imply

$$\nabla_b = E^a{}_b \nabla_a.$$

Putting $\phi_\mu{}^\lambda = D_\mu \xi^\lambda$, $\psi_\mu{}^\lambda = D_\mu \eta^\lambda$ and $\theta_\mu{}^\lambda = D_\mu \zeta^\lambda$ we have by definition of Sasakian 3-structure

$$(3.12)\quad \begin{aligned} &\phi_\mu{}^\lambda \phi_\kappa{}^\mu = -\delta_\kappa^\lambda + \xi_\kappa \xi^\lambda,\ \phi_\mu{}^\lambda \xi^\mu = 0,\ \xi_\lambda \phi_\mu{}^\lambda = 0,\ \xi_\lambda \xi^\lambda = 1,\\ &\psi_\mu{}^\lambda \psi_\kappa{}^\mu = -\delta_\kappa^\lambda + \eta_\kappa \eta^\lambda,\ \psi_\mu{}^\lambda \eta^\mu = 0,\ \eta_\lambda \psi_\mu{}^\lambda = 0,\ \eta_\lambda \eta^\lambda = 1,\\ &\theta_\mu{}^\lambda \theta_\kappa{}^\mu = -\delta_\kappa^\lambda + \zeta_\kappa \zeta^\lambda,\ \theta_\mu{}^\lambda \zeta^\mu = 0,\ \zeta_\lambda \theta_\mu{}^\lambda = 0,\ \zeta_\lambda \zeta^\lambda = 1,\\ &\theta_\mu{}^\lambda \eta^\mu = -\psi_\mu{}^\lambda \zeta_\mu = \xi^\lambda,\ \phi_\mu{}^\lambda \zeta^\mu = -\theta_\mu{}^\lambda \xi^\mu = \eta^\lambda,\ \psi_\mu{}^\lambda \xi^\mu = -\phi_\mu{}^\lambda \eta^\mu = \zeta^\lambda,\\ &\phi_\mu{}^\lambda \psi_\kappa{}^\mu = -\theta_\kappa{}^\lambda + \eta_\kappa \xi^\lambda,\ \psi_\mu{}^\lambda \theta_\kappa{}^\mu = -\phi_\kappa{}^\lambda + \zeta_\kappa \eta^\lambda,\ \theta_\mu{}^\lambda \phi_\kappa{}^\mu = \psi_\kappa{}^\lambda + \xi_\kappa \zeta^\lambda,\\ &\psi_\mu{}^\lambda \phi_\kappa{}^\mu = \theta_\kappa{}^\lambda + \xi_\kappa \eta^\lambda,\ \theta_\mu{}^\lambda \psi_\kappa{}^\mu = \phi_\kappa{}^\lambda + \eta_\kappa \zeta^\lambda,\ \phi_\mu{}^\lambda \theta_\kappa{}^\mu = \psi_\kappa{}^\lambda + \zeta_\kappa \xi^\lambda,\\ &\phi_{\mu\lambda} + \phi_{\lambda\mu} = 0,\ \psi_{\mu\lambda} + \psi_{\lambda\mu} = 0,\ \theta_{\mu\lambda} + \theta_{\lambda\mu} = 0 \end{aligned}$$

and

$$(3.13)\quad D_\mu \phi_\lambda{}^\kappa = \xi_\lambda \delta_\mu^\kappa - \xi^\kappa g_{\mu\lambda},\ D_\mu \psi_\lambda{}^\kappa = \eta_\lambda \delta_\mu^\kappa - \eta^\kappa g_{\mu\lambda},\ D_\mu \theta_\lambda{}^\kappa = \zeta_\lambda \delta_\mu^\kappa - \zeta^\kappa g_{\mu\lambda},$$

where $\xi_\kappa = \xi^\lambda g_{\lambda\kappa}$, $\eta_\kappa = \eta^\lambda g_{\lambda\kappa}$, $\zeta_\kappa = \zeta^\lambda g_{\lambda\kappa}$, $\phi_{\mu\lambda} = \phi_\mu{}^\kappa g_{\kappa\lambda}$, $\psi_{\mu\lambda} = \psi_\mu{}^\kappa g_{\kappa\lambda}$ and $\theta_{\mu\lambda} = \theta_\mu{}^\kappa g_{\kappa\lambda}$ (see Kuo [6]).

We now put in $\tilde{U}$

$$\phi_j{}^i = \phi_\mu{}^\lambda E^\mu{}_j E_\lambda{}^i,\ \psi_j{}^i = \psi_\mu{}^\lambda E^\mu{}_j E_\lambda{}^i,\ \theta_j{}^i = \theta_\mu{}^\lambda E^\mu{}_j E_\lambda{}^i.$$

Then from (3.12) we have

$$(3.14)\quad \begin{aligned} &\phi_h{}^i \phi_j{}^h = -\delta_j^i,\ \psi_h{}^i \psi_j{}^h = -\delta_j^i,\ \theta_h{}^i \theta_j{}^h = -\delta_j^i,\\ &\phi_h{}^i \psi_j{}^h = -\psi_h{}^i \phi_j{}^h = \theta_j{}^i,\ \psi_h{}^i \theta_j{}^h = -\theta_h{}^i \psi_j{}^h = \phi_j{}^i,\ \theta_h{}^i \phi_j{}^h = -\phi_h{}^i \theta_j{}^h = \psi_j{}^i. \end{aligned}$$

By using (3.8), (3.12) and (3.13), we also have

$$(3.15)\quad \begin{aligned} &\mathcal{L}_\xi \phi_j{}^i = 0,\ \mathcal{L}_\eta \phi_j{}^i = -2\theta_j{}^i,\ \mathcal{L}_\zeta \phi_j{}^i = 2\psi_j{}^i,\\ &\mathcal{L}_\xi \psi_j{}^i = 2\theta_j{}^i,\ \mathcal{L}_\eta \psi_j{}^i = 0,\ \mathcal{L}_\zeta \psi_j{}^i = -2\phi_j{}^i,\\ &\mathcal{L}_\xi \theta_j{}^i = -2\psi_j{}^i,\ \mathcal{L}_\eta \theta_j{}^i = 2\phi_j{}^i,\ \mathcal{L}_\zeta \theta_j{}^i = 0, \end{aligned}$$

$\mathcal{L}_\xi$ denoting the Lie derivation with respect to ξ, and

$$(3.16)\quad A_{ji}{}^s = -(a^s \phi_{ji} + b^s \psi_{ji} + c^s \theta_{ij}),$$

where $\phi_{ij} = \phi_j{}^h g_{hi}$, $\psi_{ji} = \psi_j{}^h g_{hi}$ and $\theta_{ji} = \theta_j{}^h g_{hi}$.

Let's denote by $K_{\kappa\mu\nu}{}^\lambda$ and $K_{kji}{}^h$ components of the curvature tensors of $(S^{4m+3}, g_{\lambda\mu})$ and (QP^m, g_{ji}) respectively. Since the unit sphere S^{4m+3} is a space of constant curvature 1, using the equation of co-Gauss we have

$$K_{kji}{}^h = K_{\kappa\mu\nu}{}^\lambda E^\kappa{}_k E^\mu{}_j E^\nu{}_i E_\lambda{}^h + A_k{}^h{}_s A_{ji}{}^s - A_j{}^h{}_s A_{ki}{}^s - 2A_{kj}{}^s A_i{}^h{}_s$$

and (3.16) implies

$$\begin{aligned} K_{kji}{}^h = &\delta_k^h g_{ji} - \delta_j^h g_{ki} + F_k{}^h F_{ji} - F_j{}^h F_{ki} - 2F_{kj} F_i{}^h + G_k{}^h G_{ji} - G_j{}^h G_{ki}\\ &- 2G_{kj} G_i{}^h + H_k{}^h H_{ji} - H_j{}^h H_{ki} - 2H_{kj} H_i{}^h. \end{aligned}$$

Hence QP^m is a quaternionic Kaehlerian manifold with constant Q-sectional curvature 4.

From now on we assume that the submanifold M is generic in the quaternionic projective space QP^m. Then, as already shown in section 2, in each coordinate neighborhood U we put

(3. 17) $$F_i{}^jB_a{}^i=\phi_a{}^bB_b{}^j+\phi_a{}^xN_x{}^j,\quad F_h{}^jN_x{}^h=-\phi_x{}^aB_a{}^j,$$
$$G_i{}^jB_a{}^i=\psi_a{}^bB_b{}^j+\psi_a{}^xN_x{}^j,\quad G_h{}^jN_x{}^h=-\psi_x{}^aB_a{}^j,$$
$$H_i{}^jB_a{}^i=\theta_a{}^bB_b{}^j+\theta_a{}^xN_x{}^j,\quad H_h{}^jN_x{}^h=-\theta_x{}^aB_a{}^j$$

and these local tensor fields $\phi_a{}^b$'s, $\phi_x{}^a$'s and $\phi_a{}^x$'s satisfy (2. 6), (2. 7) and (2. 9)-(2. 11) which will be used in the sequel.

We denote

$$\phi_\alpha{}^\beta=\phi_a{}^bE_\alpha{}^aE_b{}^\beta+(\zeta_\alpha\eta^\beta-\eta_\alpha\zeta^\beta),$$
$$\psi_\alpha{}^\beta=\psi_a{}^bE_\alpha{}^aE_b{}^\beta+(\xi_\alpha\eta^\beta-\zeta_\alpha\xi^\beta),$$
$$\theta_\alpha{}^\beta=\theta_a{}^bE_\alpha{}^aE_b{}^\beta+(\eta_\alpha\xi^\beta-\xi_\alpha\eta^\beta)$$

and

(3. 19) $$\phi_\alpha{}^x=\phi_a{}^xE_\alpha{}^a,\quad \psi_\alpha{}^x=\psi_a{}^xE_\alpha{}^a,\quad \theta_\alpha{}^x=\theta_a{}^xE_\alpha{}^a,$$
$$\phi_x{}^\alpha=\phi_x{}^aE_a{}^\alpha,\quad \psi_x{}^\alpha=\psi_x{}^aE_a{}^\alpha,\quad \theta_x{}^\alpha=\theta_x{}^aE_a{}^\alpha,$$

where we have used the same letter instead of the lift of a function.

On the other hand, from the construction of $\{F, G, H\}$ we can see that

(3. 20) $$\phi_\mu{}^\kappa E^\mu{}_j=F_j{}^hE^k{}_h,\quad \psi_\mu{}^\kappa E^\mu{}_j=G_j{}^hE^\kappa{}_h,\quad \theta_\mu{}^\kappa E_j{}^\mu=H_j{}^hE^\kappa{}_h.$$

Then we can also have

(3. 21) $$\phi_\mu{}^\kappa B_h{}^\mu=\phi_\beta{}^\alpha B_\alpha{}^\kappa+\phi_\beta{}^xN_x{}^\kappa,\quad \phi_\mu{}^\kappa N_x{}^\mu=-\phi_x{}^\alpha B_\alpha{}^\kappa,$$
$$\psi_\mu{}^\kappa B_\beta{}^\mu=\psi_\beta{}^\alpha B_\alpha{}^\kappa+\psi_\beta{}^xN_x{}^\kappa,\quad \psi_\mu{}^\kappa N_x{}^\mu=-\psi_x{}^\alpha B_\alpha{}^\kappa,$$
$$\theta_\mu{}^\kappa B_\beta{}^\mu=\theta_\beta{}^\alpha B_\alpha{}^\kappa+\theta_\beta{}^xN_x{}^\kappa,\quad \theta_\mu{}^\kappa N_x{}^\mu=-\theta_x{}^\alpha B_\alpha{}^\kappa,$$

from which and (3. 21), using (3. 12) in the usual way, we can easily get

(3. 22) $$\phi_\alpha{}^\gamma\phi_\beta{}^\alpha=-\delta_\beta{}^\gamma+\xi_\beta\xi^\gamma+\phi_\beta{}^x\phi_x{}^\gamma,\quad \phi_\beta{}^\alpha\phi_\alpha{}^x=0,$$
$$\psi_\alpha{}^\gamma\psi_\beta{}^\alpha=-\delta_\beta{}^\gamma+\eta_\beta\eta^\gamma+\psi_\beta{}^x\psi_x{}^\gamma,\quad \psi_\beta{}^\alpha\psi_\alpha{}^x=0,$$
$$\theta_\alpha{}^\gamma\theta_\beta{}^\alpha=-\delta_\beta{}^\gamma+\zeta_\beta\zeta^\gamma+\theta_\beta{}^x\theta_x{}^\gamma,\quad \theta_\beta{}^\alpha\theta_\alpha{}^x=0,$$
$$\phi_\alpha{}^\gamma\psi_\beta{}^\alpha=-\theta_\beta{}^\gamma+\eta_\beta\xi^\gamma+\psi_\beta{}^x\phi_x{}^\gamma,\quad \phi_\alpha{}^x\psi_\beta{}^\alpha=-\theta_\beta{}^x,$$
$$\psi_\alpha{}^\gamma\theta_\beta{}^\alpha=-\phi_\beta{}^\gamma+\zeta_\beta\eta^\gamma+\theta_\beta{}^x\psi_x{}^\gamma,\quad \psi_\alpha{}^x\theta_\beta{}^\alpha=-\phi_\beta{}^x,$$
$$\theta_\alpha{}^\gamma\phi_\beta{}^\alpha=\psi_\beta{}^\gamma+\xi_\beta\zeta^\gamma+\phi_\beta{}^x\theta_x{}^\gamma,\quad \theta_\alpha{}^x\phi_\beta{}^\alpha=\psi_\beta{}^x,$$
$$\phi_x{}^\alpha\phi_\alpha{}^y=\delta_x{}^y,\quad \psi_x{}^\alpha\psi_\alpha{}^y=\delta_x{}^y,\quad \theta_x{}^\alpha\theta_\alpha{}^y=\delta_x{}^y,$$
$$\phi_\beta{}^\alpha\xi^\beta=0,\quad \phi_\beta{}^\alpha\eta^\beta=-\zeta^\alpha,\quad \phi_\beta{}^\alpha\zeta^\beta=\eta^\alpha,\quad \phi_\beta{}^x\xi^\beta=0,$$
$$\psi_\beta{}^\alpha\xi^\beta=\zeta^\alpha,\quad \psi_\beta{}^\alpha\eta^\beta=0,\quad \psi_\beta{}^\alpha\zeta^\beta=-\xi^\alpha,\quad \psi_\beta{}^x\eta^\beta=0,$$
$$\theta_\beta{}^\alpha\xi^\beta=-\eta^\alpha,\quad \theta_\beta{}^\alpha\eta^\beta=\xi^\alpha,\quad \theta_\beta{}^\alpha\zeta^\beta=0,\quad \theta_\beta{}^x\zeta^\beta=0,$$
$$\phi_\beta{}^x\eta^\beta=\phi_\beta{}^x\zeta^\beta=0,\quad \psi_\beta{}^x\xi^\beta=0,\quad \theta_\beta{}^x\xi^\beta=\theta_\beta{}^x\eta^\beta=0.$$

Applying $\nabla_\gamma=B_\gamma{}^\kappa D_\kappa$ to (3. 21) and using (3. 3), (3. 10), (3. 13) and those equations, we can also easily find

(3. 23) $$\nabla_\gamma\phi_\beta{}^\alpha=\xi_\beta\delta_\gamma{}^\alpha-\xi^\alpha g_{\gamma\beta}-h_{\gamma\beta}{}^x\phi_x{}^\alpha+h_\gamma{}^\alpha{}_x\phi_\beta{}^x,\ \nabla_\gamma\xi^\alpha=\phi_\gamma{}^\alpha,$$
$$\nabla_\gamma\psi_\beta{}^\alpha=\eta_\beta\delta_\gamma{}^\alpha-\eta^\alpha g_{\gamma\beta}-h_{\gamma\beta}{}^x\psi_x{}^\alpha+h_\gamma{}^\alpha{}_x\psi_\beta{}^x,\quad \nabla_\gamma\eta^\alpha=\psi_\gamma{}^\alpha,$$
$$\nabla_\gamma\theta_\beta{}^\alpha=\zeta_\beta\delta_\gamma{}^\alpha-\zeta^\alpha g_{\gamma\beta}-h_{\gamma\beta}{}^x\theta_x{}^\alpha+h_\gamma{}^\alpha{}_x\theta_\beta{}^x,\quad \nabla_\gamma\zeta^\alpha=\theta_\gamma{}^\alpha,$$
$$\nabla_\gamma\phi_\beta{}^x=-h_{\gamma\alpha}{}^x\phi_\beta{}^\alpha,\quad \nabla_\gamma\psi_\beta{}^x=-h_{\gamma\alpha}{}^x\psi_\beta{}^\alpha,\quad \nabla_\gamma\theta_\beta{}^x=-h_{\gamma\alpha}{}^x\theta_\beta{}^\alpha,$$
$$h_{\beta\alpha}{}^x\xi^\alpha=\phi_\beta{}^x,\quad h_{\beta\alpha}{}^x\eta^\alpha=\psi_\beta{}^x,\quad h_{\beta\alpha}{}^x\zeta^\alpha=\theta_\beta{}^x,$$
$$h_{\gamma\alpha}{}^y\phi_x{}^\alpha=h_\gamma{}^\alpha{}_x\phi_\alpha{}^y,\quad h_{\gamma\alpha}{}^y\psi_x{}^\alpha=h_\gamma{}^\alpha{}_x\psi_\alpha{}^y,\quad h_{\gamma\alpha}{}^y\theta_x{}^\alpha=h_\gamma{}^\alpha{}_x\theta_\alpha{}^y.$$

Moreover, in such a manifold $\bar{M}$, equations of Gauss, Codazzi and Ricci are respectively given by

(3.24)
$$K_{\delta\gamma\beta}{}^{\alpha}=\delta_{\delta}{}^{\alpha}g_{\gamma\beta}-\delta_{\gamma}{}^{\alpha}g_{\delta\beta}+h_{\delta}{}^{\alpha}{}_{x}h_{\gamma\beta}{}^{x}-h_{\gamma}{}^{\alpha}{}_{x}h_{\delta\beta}{}^{x},$$
$$\nabla_{\gamma}h_{\beta\alpha}{}^{x}-\nabla_{\beta}h_{\gamma\alpha}{}^{x}=0,$$
$$K_{\beta\alpha y}{}^{x}=h_{\beta\gamma}{}^{x}h_{\alpha}{}^{\gamma}{}_{y}-h_{\alpha\gamma}{}^{x}h_{\beta}{}^{\gamma}{}_{y},$$

because the ambient manifold S^{4m+3} is a space of constant curvature 1.

We now come back to (3.2) and apply the operator $\nabla_{b}=B_{b}{}^{i}\nabla_{i}=E^{\beta}{}_{b}\nabla_{\beta}$ to this equation. Then we have
$$(\nabla_{b}B_{a}{}^{j})E_{\alpha}{}^{a}+B_{a}{}^{j}E^{\beta}{}_{b}\nabla_{\beta}E_{\alpha}{}^{a}=B_{b}{}^{i}E^{\mu}{}_{i}(D_{\mu}E_{\kappa}{}^{j})B_{\alpha}{}^{\kappa}+E_{\kappa}{}^{j}E^{\beta}{}_{b}\nabla_{\beta}B_{\alpha}{}^{\kappa},$$
which and (3.8) (3.11) imply
$$(h_{ba}{}^{x}E_{\alpha}{}^{a})N_{x}{}^{j}+(A_{b}{}^{a}{}_{s}C_{\alpha}{}^{s})B_{a}{}^{j}=A_{i}{}^{i}{}_{s}B_{b}{}^{i}C_{\alpha}{}^{s}+h_{\beta\alpha}{}^{x}E^{\beta}{}_{b}N_{x}{}^{j},$$
from which, taking the normal part, we have

(3.25)
$$h_{\beta\alpha}{}^{x}E^{\beta}{}_{b}=h_{ba}{}^{x}E_{\alpha}{}^{a}+\xi_{\alpha}\phi_{b}{}^{x}+\eta_{\alpha}\psi_{b}{}^{x}+\zeta_{\alpha}\theta_{b}{}^{x}$$

with the help of (3.16) and (3.17). Transvecting the above equation with $E^{\gamma}{}_{b}$ and changing the index γ with β, we get

(3.26)
$$h_{\beta\alpha}{}^{x}=h_{ba}{}^{x}E_{\beta}{}^{b}E_{\alpha}{}^{a}+\phi_{\beta}{}^{x}\xi_{\alpha}+\psi_{\beta}{}^{x}\eta_{\alpha}+\theta_{\beta}{}^{x}\zeta_{\alpha}+\phi_{\alpha}{}^{x}\xi_{\beta}+\psi_{\alpha}{}^{x}\eta_{\beta}+\theta_{\alpha}{}^{x}\zeta_{\beta}$$

with the help of (3.19), from which, taking account of (3.22),
$$g^{\beta\alpha}h_{\beta\alpha}{}^{x}=g^{ba}h_{ba}{}^{x}.$$
Thus from (3.26) we have

LEMMA 3.1. (Y. Shibuya [14]) *The mean curvature of $\bar{M}$ is the same as that of M.*

From now on, we write $h_{e}{}^{ex}$ and $h_{\alpha}{}^{\alpha x}$ as the same letter h^{x}. Moreover, the mean curvature vector of $\bar{M}$ is given by
$$H^{\kappa}=\frac{1}{n+1}h^{x}C_{x}{}^{\kappa}.$$
If $\nabla_{\beta}h^{x}=0$, the mean curvature vector H is said to be parallel in the normal bundle of $\bar{M}$. Hence, as a direct consequence of Lemma 2.1, we obtain

LEMMA 3.2. (Y. Shibuya [14]) *The mean curvature vector defined on $\bar{M}$ is parallel in the normal bundle if and only if so is the mean curvature vector on M.*

Moreover, transvecting $h_{\gamma}{}^{\alpha}{}_{y}$ to (3.26) and using (3.19), (3.22) and (3.25), we have

(3.27)
$$\begin{aligned}h_{\beta\alpha}{}^{x}h_{\gamma}{}^{\alpha}{}_{y}=&(h_{ba}{}^{x}h_{c}{}^{a}{}_{y}+\phi_{b}{}^{x}\phi_{cy}+\psi_{b}{}^{x}\psi_{cy}+\theta_{b}{}^{x}\theta_{cy})E_{\beta}{}^{b}E_{\gamma}{}^{c}+h_{ba}{}^{x}\phi_{y}{}^{a}E_{\beta}{}^{b}\xi_{\gamma}\\&+h_{b}{}^{a}{}_{y}\psi_{a}{}^{x}\eta_{\beta}E_{\gamma}{}^{b}+h_{ba}{}^{x}\psi_{y}{}^{a}E_{\beta}{}^{b}\eta_{\gamma}+h_{b}{}^{a}{}_{y}\theta_{a}{}^{x}\zeta_{\beta}E_{\gamma}{}^{b}\\&+h_{ba}{}^{x}\theta_{y}{}^{a}E_{\beta}{}^{b}\zeta_{\gamma}+h_{b}{}^{a}{}_{y}\phi_{a}{}^{x}\xi_{\beta}E_{\gamma}{}^{b},\end{aligned}$$

which and (3.23) imply

(3.28)
$$K_{\beta\gamma y}{}^{x}=K_{bcy}{}^{x}E_{\beta}{}^{b}E_{\gamma}{}^{c}.$$

Thus we have

LEMMA 3.3. (Y. Shibuya [14]) *In other that the connection in the normal bundle of $\overline{M}$ in S^{4m+3} is flat, it is necessary and sufficient that the connection in the normal bundle of M in QP^m is flat.*

In our further considerations in this section we assume that the structure tensors $\{\phi, \psi, \theta\}$ commute with the second fundamental tensor of $M(\subset QP^m)$ and the normal connection of M is flat, that is,

$$h_{ce}{}^x\phi_b{}^e+h_{be}{}^x\phi_c{}^e=0, \tag{3.29}$$

$$h_{ce}{}^x\psi_b{}^e+h_{be}{}^x\psi_c{}^e=0, \tag{3.30}$$

$$h_{ce}{}^x\theta_b{}^e+h_{be}{}^x\theta_c{}^e=0 \tag{3.31}$$

and

$$h_{de}{}^x h_c{}^e{}_y-h_{ce}{}^x h_d{}^e{}_y+\phi_d{}^x\phi_{cy}-\phi_c{}^x\phi_{dy}+\psi_d{}^x\psi_{cy}-\psi_c{}^x\psi_{dy}+\theta_d{}^x\theta_{cy}-\theta_c{}^x\theta_{dy}=0, \tag{3.32}$$

with the help of (2.20) with $c=4$.

Transvecting $\phi_a{}^b$ to (3.29), we obtain

$$h_{ca}{}^x-h_{ce}{}^x\phi_a{}^y\phi_y{}^e+h_{be}{}^x\phi_c{}^e\phi_a{}^b=0,$$

from which, taking the skew-symmetric part, $h_{ce}{}^x\phi_a{}^y\phi_y{}^e-h_{ae}{}^x\phi_c{}^y\phi_y{}^e=0$. Transvecting with $\phi_z{}^c$ yields

$$h_{ae}{}^x\phi_z{}^e=P_{yz}{}^x\phi_a{}^y, \tag{3.33}$$

where we put $h_{ce}{}^x\phi_y{}^e\phi_z{}^c=P_{yz}{}^x$. Similarly, we can put $h_{ce}{}^x\psi_y{}^e\psi_z{}^c=Q_{yz}{}^x$ and $h_{ce}{}^x\theta_y{}^e\theta_z{}^c=R_{yz}{}^x$. Putting $P_{yzx}=P_{yz}{}^w g_{wx}$, we see that P_{yzx} is symmetric for any index because of (3.24) and the definition of $P_{yz}{}^x$.

From (3.32), transvecting with $\phi_z{}^c$ and using (3.33), we have

$$P_{yz}{}^u P_{uw}{}^x\phi_d{}^w-P_{zu}{}^x P_{yw}{}^u\phi_d{}^w=\phi_d{}^x g_{zy}-\phi_{dy}\delta_z{}^x,$$

or, transvecting with $\phi_v{}^d$ this yields

$$P_{yz}{}^u P_{uw}{}^x-P_{zu}{}^x P_{yw}{}^u=\delta_w{}^x g_{zy}-g_{wy}\delta_z{}^x,$$

from which, we obtain

$$P_{zux}P_w{}^{zu}=P^u P_{uwx}-(p-1)g_{wx}, \tag{3.34}$$

where p is the codimension of M and

$$P^x=g^{yz}P_{yz}{}^x. \tag{3.35}$$

On the other hand, if a generic submanifold of QP^m with flat normal connection satisfies (3.29), (3.30) and (3.31), then we can easily obtain

$$h_{\beta\alpha}{}^x h_\gamma{}^\alpha{}_y=P_{yz}{}^x h_{\beta\gamma}{}^z+g_{\beta\gamma}\delta_y{}^x. \tag{3.36}$$

Moreover, by a simple computation we find

$$h_{\beta\alpha}{}^x\phi_\gamma{}^\alpha+h_{\gamma\alpha}{}^x\phi_\beta{}^\alpha=0, \tag{3.37}$$

$$h_{\beta\alpha}{}^x\psi_\gamma{}^\alpha+h_{\gamma\alpha}{}^x\psi_\beta{}^\alpha=0, \tag{3.38}$$

$$h_{\beta\alpha}{}^x\theta_\gamma{}^\alpha+h_{\gamma\alpha}{}^x\theta_\beta{}^\alpha=0 \tag{3.39}$$

with the help of (3.22), (3.26) and (3.29)-(3.31). Transvecting (3.26) with $\phi_y{}^\alpha$ this yields

(3.40) $$h_{\beta\alpha}{}^x\phi_y{}^\alpha=P_{yz}{}^x\phi_\beta{}^z+\delta_y{}^x\xi_\beta$$

with the aid of (3.18) and (3.33), from which, transvecting $\phi_w{}^\beta$, we find

(3.41) $$P_{yz}{}^x=h_{\beta\alpha}{}^x\phi_y{}^\alpha\phi_z{}^\beta$$

or, transvecting g^{yz} this yields

(3.41) $$P^x=h_{\beta\alpha}{}^x\phi_y{}^\beta\phi^{y\alpha}.$$

Now, differentiating (3.37) covariantly and using (3.23), we obtain

$$(\nabla_\delta h_{\beta\alpha}{}^x)\phi_\gamma{}^\alpha+h_{\beta\alpha}{}^x(\xi_\gamma\delta_\delta{}^\alpha-\xi^\alpha g_{\delta\gamma}-h_{\delta\gamma}{}^y\phi_y{}^\alpha+h_\delta{}^\alpha{}_y\phi_\gamma{}^y)$$
$$+(\nabla_\delta h_{\gamma\alpha}{}^x)\phi_\beta{}^\alpha+h_{\gamma\alpha}{}^x(\xi_\beta\delta_\delta{}^\alpha-\xi^\alpha g_{\delta\beta}-h_{\delta\beta}{}^y\phi_y{}^\alpha+h_\delta{}^\alpha{}_y\phi_\beta{}^y)=0,$$

and then, using (3.36) and (3.40), we find

$$(\nabla_\delta h_{\beta\alpha}{}^x)\phi_\gamma{}^\alpha+(\nabla_\delta h_{\gamma\alpha}{}^x)\phi_\beta{}^\alpha=0,$$

from which, taking the skew-symmetric part with respect to the indices δ and β,

$$(\nabla_\delta h_{\gamma\alpha}{}^x)\phi_\beta{}^\alpha=\nabla_\beta h_{\gamma\alpha}{}^x\phi_\delta{}^\alpha.$$

Since the ambient manifold S^{4m+3} is a space of constant curvature 1. Hence, the last two equations imply $(\nabla_\gamma h_{\beta\alpha}{}^x)\phi_\delta{}^\alpha=0$, from which, transvecting with $\phi_\varepsilon{}^\delta$, we obtain

$$\nabla_\gamma h_{\beta\varepsilon}{}^x=(\nabla_\gamma h_{\beta\alpha}{}^x)\xi_\varepsilon\xi^\alpha+(\nabla_\gamma h_{\beta\alpha}{}^x)\phi_\varepsilon{}^y\phi_y{}^\alpha,$$

by means of (3.22). Contracting with the indices β and ε gives

$$\nabla_\gamma h^x=(\nabla_\gamma h_{\beta\alpha}{}^x)\xi^\beta\xi^\alpha+(\nabla_\gamma h_{\beta\alpha}{}^x)\phi^{\beta y}\phi_y{}^\alpha.$$

By a simple computation, we can easily obtain

$$(\nabla_\gamma h_{\beta\alpha}{}^x)\xi^\beta\xi^\alpha=0.$$

Consequently, we have

(3.43) $$\nabla_\gamma h^x=(\nabla_\gamma h_{\beta\alpha}{}^x)\phi^{\beta x}\phi_x{}^\alpha.$$

Differentiating (3.42) covariantly and using (3.43), we find

$$\nabla_\gamma P^x=\nabla_\gamma h^x-(h_{\beta\alpha}{}^x\phi^{y\alpha})(h_\gamma{}^\varepsilon{}_y\phi_\varepsilon{}^\beta)-(h_{\beta\alpha}{}^x\phi_y{}^\beta)(h_{\gamma\varepsilon}{}^y\phi^{\varepsilon\alpha}),$$

or, substituting (3.40) into the above equation, we have

(3.44) $$\nabla_\gamma h^x=\nabla_\gamma P^x.$$

Thus we obtain

LEMMA 3.4. *Let M be a generic submanifold of QP^m with flat normal connection. If M satisfies*

$$h_{ce}{}^x\phi_b{}^e+h_{be}{}^x\phi_c{}^e=0,$$
$$h_{ce}{}^x\psi_b{}^e+h_{be}{}^x\psi_c{}^e=0,$$
$$h_{ce}{}^x\theta_b{}^e+h_{be}{}^x\theta_c{}^e=0,$$

then we have

(3.45) $$\nabla_\gamma h^x=\nabla_\gamma P^x.$$

Next, we prove

LEMMA 3.5. *Under the same assumptions given in Lemma* 3.4, *we have*

$$\frac{1}{2}\Delta(h_{\alpha\beta}{}^{x}h^{\alpha\beta}{}_{x})=(\nabla_{\alpha}\nabla_{\beta}h^{x})h^{\alpha\beta}{}_{x}+\|\nabla_{\gamma}h_{\beta\alpha}{}^{x}\|^{2}, \tag{3.46}$$

where Δ *is the Laplacian given by* $\Delta=g^{\gamma\beta}\nabla_{\gamma}\nabla_{\beta}$.

Proof. From the Ricci identity, we have

$$\nabla^{\gamma}\nabla_{\gamma}h_{\beta\alpha}{}^{x}-\nabla_{\beta}\nabla_{\alpha}h^{x}=K_{\beta\gamma}h_{\alpha}{}^{\gamma x}-K_{\delta\beta\alpha\gamma}h^{\delta\gamma x} \tag{3.47}$$

with the help of (3.24), where $K_{\beta\gamma}$ is the Ricci tensor given by

$$K_{\beta\gamma}=(n-2)g_{\beta\gamma}+h^{x}h_{\beta\gamma x}-h_{\beta\alpha}{}^{x}h_{\gamma}{}^{\alpha}{}_{x} \tag{3.48}$$

by virtue of (3.24). Transvecting (3.47) with $h^{\beta\alpha}{}_{x}$ and taking account of (2.24), (3.36) and (3.48), we find

$$(\nabla^{\gamma}\nabla_{\gamma}h_{\beta\alpha}{}^{x})h^{\beta\alpha}{}_{x}-(\nabla_{\beta}\nabla_{\alpha}h^{x})h^{\beta\alpha}{}_{x}=0$$

with the help of (3.34). Therefore, we have

$$\frac{1}{2}\Delta(h_{\beta\alpha}{}^{x}h^{\beta\alpha}{}_{x})=(\nabla_{\beta}\nabla_{\alpha}h^{x})h^{\beta\alpha}{}_{x}+\|\nabla_{\gamma}h_{\beta\alpha}{}^{x}\|^{2}.$$

Thus we completes the proof of the lemma.

On the other hand, from (3.36) we obtain

$$h_{\beta\alpha}{}^{x}h^{\beta\alpha}{}_{x}=h^{x}P_{x}+(n+3)p.$$

Hence from (3.19), we have

$$\|\nabla_{\gamma}h_{\beta\alpha}{}^{x}\|^{2}=0,$$

that is,

$$\nabla_{\gamma}h_{\beta\alpha}{}^{x}=0.$$

Combining with Theorem B in section 1, we find

THEOREM 3.6. *Let M be an n-dimensional complete, generic submanifold of a quaternionic projective space* QP^{m} *with flat normal connection. If the second fundamental tensors* $h_{ba}{}^{x}$ *on M commute with the structure tensors* ϕ, ψ *and,* θ i.e.,

$$h_{be}{}^{x}\phi_{c}{}^{e}+h_{ce}{}^{x}\phi_{b}{}^{e}=0,$$
$$h_{be}{}^{x}\psi_{c}{}^{e}+h_{ce}{}^{x}\psi_{b}{}^{e}=0,$$
$$h_{be}{}^{x}\theta_{c}{}^{e}+h_{ce}{}^{x}\theta_{b}{}^{e}=0,$$

and if the mean curvature vector defined on M is parallel in the normal bundle, then M is of the form

$$\tilde{\pi}(S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N)),$$

where $p_1,\cdots,p_N\geqq 1$, $p_i=4l_i+3$ (l_i: *non-negative integer*), $p_1+\cdots+p_N=n+3$, $\sum_{i=1}^{N}r_i^2=1$, $N=4m-n+1$.

4. An integral formula

In this section we also assume that the submanifold of QP^m is generic.

It is well know (Ishihara [2]) that for a quaternionic projective space $QP^m (m \geqq 2)$ with constant Q-sectional curvature $c=4$, the following are valid:

$$D_j p_i - D_i p_j + q_j r_i - r_j q_i = 4F_{ji},$$
$$D_j q_i - D_i q_j + r_j p_i - p_j r_i = -4G_{ji},$$
$$D_j r_i - D_i r_j + p_j q_i - q_j p_i = -4H_{ji}.$$

Therefore, in a submanifold M of QP_m the local 1-forms p, q, r definied by

$$p_b = p_i B_b{}^i, \quad q_b = q_i B_b{}^i, \quad r_b = r_i B_b{}^i$$

satisfy

(4.1)
$$\nabla_b p_a - \nabla_a p_b + q_b r_a - r_b q_a = -4\phi_{ba},$$
$$\nabla_b q_a - \nabla_a q_b + r_b p_a - p_b r_a = -4\psi_{ba},$$
$$\nabla_b r_a - \nabla_a r_b + p_b q_a - q_b p_a = -4\theta_{ba}.$$

On the other hand, taking account of arguments developed in section 2, we see easily that there are two global vector fields S_1 and S_2 on M with components

$$\phi_x{}^e (\mathring{\nabla}_e \phi_{xb}) + \psi_x{}^e (\mathring{\nabla}_e \psi^{xb}) + \theta_x{}^e (\mathring{\nabla}_e \theta^{xb}),$$
$$(\mathring{\nabla}_e \phi_x{}^e) \phi^{xb} + (\mathring{\nabla}_e \psi_x{}^e) \psi^{xb} + (\mathring{\nabla}_e \theta_x{}^e) \theta^{xb},$$

respectively. Putting

$$\mathring{\nabla}_c \mathring{\nabla}_b \phi_a{}^x = \nabla_c \mathring{\nabla}_b \phi_a{}^x - r_c \mathring{\nabla}_b \psi_a{}^x + q_c \mathring{\nabla}_b \theta_a{}^x,$$
$$\mathring{\nabla}_c \mathring{\nabla}_b \psi_a{}^x = \nabla_b \mathring{\nabla}_b \psi_a{}^x + r_c \mathring{\nabla}_b \phi_a{}^x - p_c \mathring{\nabla}_b \theta_a{}^x,$$
$$\mathring{\nabla}_c \mathring{\nabla}_b \theta_a{}^x = \nabla_c \mathring{\nabla}_b \theta_a{}^x - q_c \mathring{\nabla}_b \phi_a{}^x + p_c \mathring{\nabla}_b \psi_a{}^x,$$

and taking account of (4.1), we can verify

$$\mathring{\nabla}_c \mathring{\nabla}_b \phi_a{}^x - \mathring{\nabla}_b \mathring{\nabla}_c \phi_a{}^x = -K_{cba}{}^e \phi_e{}^x + K_{cby}{}^x \phi_a{}^y + 4\theta_{cb} \psi_a{}^x - 4\psi_{cb} \theta_a{}^x,$$
$$\mathring{\nabla}_c \mathring{\nabla}_b \psi_a{}^x - \mathring{\nabla}_b \mathring{\nabla}_c \psi_a{}^x = -K_{cba}{}^e \psi_e{}^x + K_{cby}{}^x \psi_a{}^y - 4\theta_{cb} \theta_a{}^x + 4\phi_{cb} \theta_a{}^x,$$
$$\mathring{\nabla}_c \mathring{\nabla}_b \theta_a{}^x - \mathring{\nabla}_b \mathring{\nabla}_c \theta_a{}^x = -K_{cba}{}^e \theta_e{}^x + K_{cby}{}^x \theta_a{}^y + 4\psi_{cb} \phi_a{}^x - 4\phi_{cb} \psi_a{}^x,$$

which implies

$$\begin{aligned} \mathring{\nabla}_b S_1{}^b - \mathring{\nabla}_b S_2{}^b = {} & K_{ae} (\phi_x{}^e \phi^{xa} + \psi_x{}^e \psi^{xa} + \theta_x{}^e \theta^{xa}) - K_{aex}{}^y (\phi_y{}^a \phi^{xe} + \psi_y{}^a \psi^{xe} + \theta_y{}^a \theta^{xe}) \\ & -24p + (\mathring{\nabla}_b \phi_x{}^e)(\mathring{\nabla}_e \phi^{xb}) + (\mathring{\nabla}_b \psi_x{}^e)(\mathring{\nabla}_e \psi^{xb}) + (\mathring{\nabla}_b \theta_x{}^e)(\mathring{\nabla}_e \theta^{xb}) \\ & - (\|\mathring{\nabla}_e \phi_x{}^e\|^2 + \|\mathring{\nabla}_e \psi_x{}^e\|^2 + \|\mathring{\nabla}_e \theta_x{}^e\|^2), \end{aligned}$$

or equivalently

(4.2)
$$\begin{aligned} \mathring{\nabla}_b S_1{}^b - \mathring{\nabla}_b S_2{}^b = {} & K_{ae} (\phi_x{}^e \phi^{xa} + \psi^e{}_x \psi^{xa} + \theta_x{}^e \theta^{xa}) - 24p - (\mathring{\mathrm{div}}\phi)^2 + (\mathring{\mathrm{div}}\psi)^2 \\ & + (\mathring{\mathrm{div}}\theta)^2 + \frac{1}{2} \|\mathring{\mathcal{L}}_\phi g\|^2 + \|\mathring{\mathcal{L}}_\psi h\|^2 + \|\mathring{\mathcal{L}}_\theta g\|^2 \\ & - (\|\nabla_a \phi_x{}^e\|^2 + \|\nabla_a \psi_x{}^e\|^2 + \nabla_a \theta_x{}^e\|^2), \end{aligned}$$

where $\mathring{\mathcal{L}}_\phi g = \nabla_b \phi_a{}^x + \nabla_a \phi_b{}^x$, $\mathring{\mathcal{L}}_\psi g = \mathring{\nabla}_b \psi_a{}^x + \mathring{\nabla}_a \psi_b{}^x$, $\mathring{\mathcal{L}}_\theta g = \mathring{\nabla}_b \theta_a{}^x + \mathring{\nabla}_a \theta_b{}^x$ and $\mathring{\mathrm{div}}\phi = \nabla_a \phi_x{}^a$, $\mathring{\mathrm{div}}\psi = \nabla_a \psi_x{}^a$, $\mathring{\mathrm{div}}\theta = \nabla_a \theta_x{}^a$. On the other hand, (2.15), (2.16) and (2.17) give

$$\|\mathring{\nabla}_b\phi_a{}^x\|^2+\|\mathring{\nabla}_b\psi_a{}^x\|^2+\|\mathring{\nabla}_b\theta_a{}^x\|^2=3\|h_{ba}{}^x\|^2-(\|h_{be}{}^x\phi_x{}^e\|^2+\|h_{be}{}^x\psi_x{}^e\|^2+\|h_{be}{}^x\theta_x{}^e\|^2),$$
$$\mathring{\mathrm{div}}\phi=\mathring{\mathrm{div}}\psi=\mathring{\mathrm{div}}\theta=0,$$
$$K_{ae}(\phi_x{}^e\phi^{xa}+\psi^e{}_x\psi^{xa}+\theta_x{}^e\theta^{xa})+(\|h_a{}^d{}_x\phi^{ya}\|^2+\|h_a{}^d{}_y\psi^{ya}\|^2+\|h_a{}^d{}_y\theta^{ya}\|^2)$$
$$=3p(n+5)+h^yh_{aey}(\phi_x{}^e\phi^{xa}+\psi_x{}^e\psi^{xa}+\theta_x{}^e\theta^{xa}).$$

Substituting these equations in (4.2), we obtain

$$(4.3)\quad \mathring{\nabla}_bS_1{}^b-\mathring{\nabla}_bS_2{}^b=3p(n-3)+\frac{1}{2}(\|\mathring{\mathcal{L}}_\phi g\|^2+\|\mathring{\mathcal{L}}_\psi g\|^2+\|\mathring{\mathcal{L}}_\theta g\|^2)-3\|h_{ba}{}^x\|^2$$
$$-K_{aex}{}^y(\phi_y{}^a\phi^{xe}+\psi_y{}^a\psi^{xe}+\theta_y{}^a\theta^{xe})+h^yh_{aey}(\phi_x{}^e\phi^{xa}+\psi_x{}^e\psi^{xa}+\theta_x{}^e\theta^{xa}).$$

We can now prove

THEOREM 4.1. *Let M be an n-dimensional compact orientable, generic submanifold of a quaternionic projective space $QP^m(m\geqq 2)$. Then the condition*

$$\int_M\{3p(n-3)-3\|h_{ba}{}^x\|^2+h^yh_{aey}(\phi_x{}^e\phi^{xa}+\psi_x{}^e\psi^{xa}+\theta_x{}^e\theta^{xa})$$
$$-K_{aex}{}^y(\phi_y{}^a\phi^{xe}+\psi_y{}^a\psi^{xe}+\theta_y{}^a\theta^{xe})\}*1=0$$

is equivalent to

$$h_{ce}{}^x\phi_b{}^e+h_{be}{}^x\phi_c{}^e=0,$$
$$h_{ce}{}^x\psi_b{}^e+h_{be}{}^x\psi_c{}^e=0,$$
$$h_{ce}{}^x\theta_b{}^e+h_{be}{}^x\theta_c{}^e=0.$$

Combining Theorem 4.1 and Theorem A, we have

THEOREM 4.2. *Let M be an n-dimensional compact, orientable, minimal and generic submanifold of QP^m whose normal connection is flat. If the second fundamental tensors $h_{ba}{}^x$ of M satisfy*

$$h^{ba}{}_xh_{ba}{}^x\leqq p(4m-p-3)$$

at each point of M, then M is

$$\tilde{\pi}(S^{p_1}(r_1)\times\cdots\times S^{p_N}(r_N)),\quad r_t=\sqrt{p_t/n+3}\quad(t=1,\cdots,N),$$

where $n+3=\sum_{i=1}^{N}p_i$, p_i's are of the form $4l_i+3$ (l_i : non-negative integer) and $p=N-1$.

As a special case, we consider the case $p=1$. Then we have

COROLLARY 4.3. (Lawson [7]) *Let M be a compact, orientable and minimal real hypersurface in QP^m. If $h_{ba}\ h^{ba}\leqq 4(m-1)$ at each point of M, then M is*

$$\tilde{\pi}(S^k(k/n+3)\times S^{n-k}((n-k+3)/(n+3))),\quad 1\leqq k\leqq n+3.$$

On the other hand, for any vector X^a tangent to M, if $h_{ba}{}^x X^b X^a \geqq 0$ or $h_{ba}{}^x X^b X^a \leqq 0$ for all x, then the second fundamental form of M is said to be semidefinite. In this point of view one of the present authors proved in [11]

THEOREM C. *Let M be a compact real hypersurface of $QP^{(n+1)/4}$. If the second fundamental form h_{ba} is semidefinite and the mean curvature is constant and if $h_{ba}h^{ba} \leqq n-3$, then M is of the form*

$$\tilde{\pi}(S^n(r) \times S^1(r)), \quad r = \sqrt{1/2}$$

As an analogue for generic submanifolds, we have

THEOREM 4.4. *Let M be a compact n-dimensional generic submanifold of $QP^{(n+p)/4}$ with flat normal connection such that the second fundamental form is semidefinite. If $h_{ba}{}^x h^{ba}{}_x \leqq (n-3)p$, then $p=1$ and M is of the form*

$$\tilde{\pi}(S^n(r) \times S^1(r)), \quad r = \sqrt{1/2}$$

Proof. Since M is compact, (4.3) implies that

$$\int_M \{3(n-3)p - h_{ba}{}^x h^{ba}{}_x) + h^x h_{bax}(\phi_y{}^b \phi^{ya} + \psi_y{}^b \psi^{ya} + \theta_y{}^b \theta^{ya})\} *1$$
$$= -\frac{1}{2}\int_M (\|\mathring{\mathcal{L}}_\phi g\|^2 + \|\mathring{\mathcal{L}}_\psi g\|^2 + \|\mathring{\mathcal{L}}_\theta g\|^2) *1.$$

From the assumptions we see that the left hand side of this equation is nonnegative. Thus it must be that

$$h_{ba}{}^x h^{ba}{}_x = (n-3)p, \tag{4.4}$$
$$h^x h_{bax}(\phi_y{}^b \phi^{ya} + \psi_y{}^b \psi^{ya} + \theta_y{}^b \theta^{ya}) = 0, \tag{4.5}$$
$$h_{be}{}^x \phi_c{}^e + h_{ce}{}^x \phi_b{}^e = 0, \quad h_{be}{}^x \psi_c{}^e + h_{ce}{}^x \psi_b{}^e = 0, \quad h_{be}{}^x \theta_c{}^e + h_{ce}{}^x \theta_b{}^e = 0. \tag{4.6}$$

Suppose that $h^x = 0$ for some x. Since the second fundamental form is semidefinite it follows that $h_{ba}{}^x = 0$ for the index x. On the other hand, the equation of Codazzi with $h_{ba}{}^x = 0$ gives

$$\phi_b{}^a = 0, \quad \psi_b{}^a = 0, \quad \theta_b{}^a = 0.$$

Consequently

$$h_{ba}{}^x \phi_y{}^b \phi^{ya} = h^x, \quad h_{ba}{}^x \psi_y{}^b \psi^{ya} = h^x, \quad h_{ba}{}^x \theta_y{}^b \theta^{ya} = h^x,$$

from which and (4.5), we have $h^z h_z = 0$, that is, $h^z = 0$ for all z, and hence M is totally geodesic. This contradicts to the fact that $h_{ba}{}^x h^{ba}{}_x = (n-3)p$. Therefore, we must have

$$g^{yz} h_{ba}{}^x \phi_y{}^b \phi_z{}^a = 0, \quad g^{yz} h_{ba}{}^x \psi_y{}^b \psi_z{}^a = 0, \quad g^{yz} h_{ba}{}^x \theta_y{}^b \theta_z{}^a = 0.$$

Since $g^{yz} = \delta^{yz}$ and the second fundamental form is semidefinite, the above equations imply

$$h_{ba}{}^x \phi_y{}^b \phi_y{}^a = 0, \quad h_{ba}{}^x \psi_y{}^b \psi_y{}^a = 0, \quad h_{ba}{}^x \theta_y{}^b \theta_y{}^a = 0,$$

where the double indices do not mean summation. Hence we find

$$P_{yz}{}^x = h_{ba}{}^x \phi_y{}^b \phi_z{}^a = h_{ba}{}^x \psi_y{}^b \psi_z{}^a = h_{ba}{}^x \theta_y{}^b \theta_z{}^a = 0,$$

and consequently

$$h_{ba}{}^{x}\phi_{y}{}^{a}=0,\quad h_{ba}{}^{x}\psi_{y}{}^{a}=0,\quad h_{ba}{}^{x}\theta_{y}{}^{a}=0. \tag{4.7}$$

We now use the equation (2.20) with $c=4$ of Ricci. Then transvecting with $\phi_{x}{}^{c}\phi^{by}$ and using (4.7) give $p=1$. On the other hand (4.6) implies that the mean curvature of M is constant because of $p=1$ (see [11]). Thus our theorem follows from Theorem C.

From Theorem 4.4 we have

THEOREM 4.5. *Let M be a compact real hypersurface of $QP^{(n+p)/4}$ such that the second fundamental form is semidefinite. If $h_{ba}h^{ba}\leqq n-3$, then M is of the form*

$$\tilde{\pi}(S^{n}(r)\times S^{1}(r)),\quad r=\sqrt{1/2}$$

On the other hand, it is known that the second fundamental forms are commutative if

$$h_{be}{}^{x}h_{a}{}^{e}{}_{y}=h_{ae}{}^{x}h_{b}{}^{e}{}_{y}.$$

In this case the integral formula in Theorem 4.1 is reduced to

$$\int_{M}\{3(p(n-p-2)-h_{ba}{}^{x}h^{ba}{}_{x})+h^{y}h_{aey}(\phi_{x}{}^{e}\phi^{xa}+\psi_{x}{}^{e}\psi^{xa}+\theta_{x}{}^{e}\theta^{xa})\}*1$$
$$=-\frac{1}{2}\int_{M}(\|\mathring{\mathcal{L}}_{\phi}g\|^{2}+\|\mathring{\mathcal{L}}_{\psi}g\|^{2}+\|\mathring{\mathcal{L}}_{\theta}g\|^{2})*1,$$

because

$$K_{aexy}(\phi_{y}{}^{a}\phi^{xe}+\psi_{y}{}^{a}\psi^{xe}+\theta_{y}{}^{a}\theta^{xe})=3p(p-1).$$

Hence, by the same way as in the proof of Theorem 4.4, we have

THEOREM 4.6. *Let M be a compact n-dimensional generic submanifold of $QP^{(n+p)/4}$. Suppose that the second fundamental forms are commutative and semidefinite. If $h_{ba}{}^{x}h^{ba}{}_{x}\leqq p(n-p-2)$, then $p=1$ and M is of the form*

$$\tilde{\pi}(S^{n}(r)\times S^{1}(r)),\quad r=\sqrt{1/2}$$

Proof. From our assumptions we can easily find by the same way as in the proof of Theorem 4.4

$$h_{ba}{}^{x}\phi_{y}{}^{a}=0,\quad h_{ba}{}^{x}\psi_{y}{}^{a}=0,\quad h_{ba}{}^{x}\theta_{y}{}^{a}=0. \tag{4.8}$$

Applying the operator ∇_{c} to the first equation of (4.8) and taking the skew-symmetric part with respect to the indices c and b, we obtain

$$-2\phi_{cb}\delta_{y}{}^{x}+\psi_{c}{}^{x}\theta_{by}-\psi_{b}{}^{x}\theta_{cy}-\theta_{c}{}^{x}\psi_{by}+\theta_{b}{}^{x}\psi_{cy}+2h_{b}{}^{ax}h_{aey}\phi_{c}{}^{e}=0,$$

where we have used (2.9), (2.11), (2.19) with $c=4$ and (4.8). Transvecting the above equation with $\phi_{d}{}^{c}$ and using (2.7) and (4.8), it follows that

$$2h_{ba}{}^{x}h_{d}{}^{a}{}_{y}=2(g_{bd}-\phi_{b}{}^{z}\phi_{zd})\delta_{y}{}^{x}-\psi_{d}{}^{x}\psi_{by}-\psi_{b}{}^{x}\psi_{dy}-\theta_{d}{}^{x}\theta_{by}-\theta_{b}{}^{x}\theta_{dy},$$

from which, transvecting with $\psi_{x}{}^{d}$, we have

$$(p-1)\psi_{by}=0$$

with the aid of $\psi_{d}{}^{x}\psi_{x}{}^{d}=p$. Hence $p=1$.

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