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SOME TOPOLOGICAL PROPERTIES OF THE LIPSCHITZ-KILLING CURVATURE

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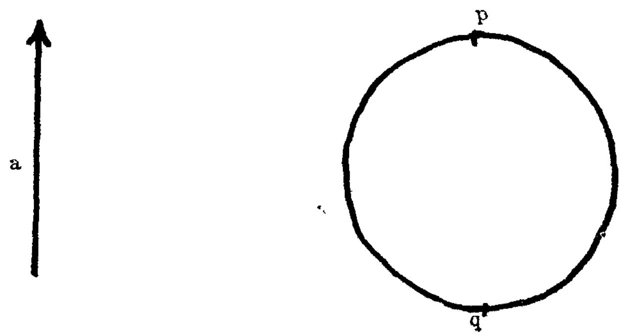
1. Introduction

In order to find relations between local properties of a submanifold and its topology, an interesting method is an application of Morse theory. We shall show, in this paper, that the sign of the Lipschitz-Killing curvature of the submanifold in a fixed direction gives important restrictions on its homology.

Let us consider the following situation: The circle S^1 in the plane $\mathbf{E}^2$.



It is clear that the Lipschitz-Killing curvature of S^1 at p and q with respect to the direction a, is positive at p and negative at q. By a suitable deformation of S^1, it is possible to find some imbedding of S^1 in $\mathbf{E}^2$ satisfying:

The curve is still the boundary of a compact set and there is at most one point q such that the Lipschitz-Killing curvature at q is strictly negative.

We shall prove the following theorem which is a generalization of this situation:

THEOREM. *Let $f: M^n \to \mathbf{E}^{n+p}$ be an isometric immersion of a compact Riemannian manifold M^n of odd dimension n, into the Euclidean space $\mathbf{E}^{n+p}$, $p \geqslant 1$.*

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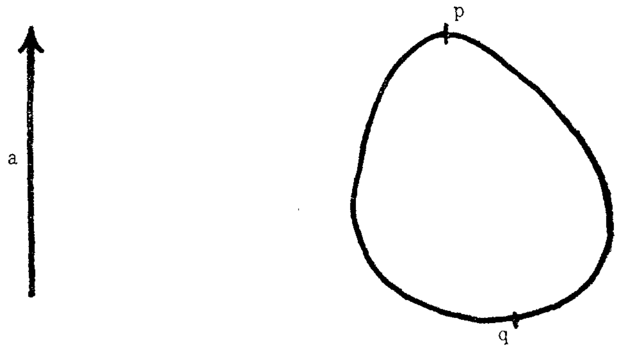
1) *If there exists a fixed vector a in* $\mathbf{E}^{n+p}$, *such that the Lipschitz-Killing curvature of* M^n *is not null at every point where a is normal to* M^n, *and positive at every point, except one, where a is normal to* M^n, *then* M^n *is an homology sphere.*

2) *If* $n \neq 3k$, $\forall k \in \mathbf{N}$, *if* $2p < n$, *and if there exists a fixed vector a in* $\mathbf{E}^{n+p}$, *such that the Lipschitz-Killing curvature of* M^n *is not null at every point where a is normal to* M^n, *and positive at every point, except at most two, where a is normal to* M^n, *then* M^n *is the boundary of a compact manifold.*



2. Notations and definitions

1) *The second fundamental form of an isometric immersion.*

Let $f : M^n \rightarrow \mathbf{E}^{n+p}$ be an isometric immersion of a Riemannian manifold M^n into the Euclidean space. We denote by $\langle \cdot \rangle$ the scalar product on $\mathbf{E}^{n+p}$ and M^n, ∇ the Levi-Civita connexion on M^n and $\tilde{\nabla}$ the trivial connexion on $\mathbf{E}^{n+p}$. TM^n and $T^{\perp}M^n$ are the tangent bundle and the normal bundle over M^n. It is well known that the second fundamental form of the immersion is the symmetric tensor $\sigma : TM^n \times TM^n \rightarrow T^{\perp}M^n$ defined by the equation

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \qquad \forall X, Y \in TM^n.$$

We shall need the following:

DEFINITION. Let $m \in M^n$, and $\xi \in T_m{}^{\perp}M^n$. The Lipschitz-Killing curvature of M^n. at m, in the direction ξ, is the determinant of the symmetric bilinear form $\langle \sigma(\cdot, \cdot), \xi \rangle$.

2) *The height function on a submanifold.*

We suppose now that M^n is compact. Let $\vec{x}$ be the position vector of M^n

in $\mathbf{E}^{n+p}$. If a is a fixed vector of $\mathbf{E}^{n+p}$, we can consider the height funtion $h_a=\langle x, a\rangle$. It is well known that:

(i) A critical point m of h_a is a point m such that $\langle X_m, a\rangle=0$, $\forall X_m\in T_mM^n$.

(ii) At a critical point, the hessian of h_a is given by

$$d^2h_a(X, Y)=\langle\sigma(X, Y), a\rangle.$$

(iii) For almost every a, h_a has non-degenerate critical points. Then h_a is a Morse function, in the case where M^n is compact.

Using the Morse inequalities (*cf.* [1]), we have, in this case : $\beta_k\leqslant\tau_k$, where β_k is the k-th Betti number of M^n (i. e., $\beta_k=\dim H_k(M^n, F)$, where $H_k(M^n, F)$ is the k-th homology group of M^n over any field F) and τ_k is the number of critical points of index k.

3) *The Stiefel Whitney numbers of a manifold* (*cf.* [2]).

Let $H^k(M^n, \mathbf{Z}/2\mathbf{Z})$ will denote the k-th cohomology group of M^n, with coefficient in $\mathbf{Z}/2\mathbf{Z}$.

Let ω_k will denote the k-th Stiefel-Whitney class of M^n. And $\omega=1+\omega_1+\cdots+\omega_n$ is the total Stiefel-Whitney class of M^n. We denote by $\bar{\omega}=1+\bar{\omega}_1+\cdots+\bar{\omega}_n$ the inverse of ω.

A Stiefel-Whitney number N is defined by the following:

$$N=\omega_1{}^{r_1}\omega_2{}^{r_2}\cdots\omega_n{}^{r_n}, \text{ with } 1r_1+\cdots+nr_n=n,$$

We recall now the well known theorem of Thom (*cf.* [2]).

THEOREM (Thom.) *Let M^n be a compact manifold. If all the Stiefel-Whitney numbers of M^n are null, then M^n is the boundary of a compact manifold.*

We shall use this theorem in the proof of our result.

3. Proof of the theorem

Since M^n is compact, there exists at least one point q on M^n such that the height function h_a has a maximum value at q. At q, we have:

$$\begin{cases} d\, h_{a_q}=0 \\ d^2h_{a_q}(X, X)\leqslant 0, \quad \forall X\in T_qM^n. \end{cases}$$

On the other hand, the Lipschitz-Killing curvature at q is not null, by assumption. Then.

$$\det \langle\sigma(\cdot, \cdot), a\rangle_q=\det d^2h_{a_q}<0.$$

We shall now examine the two different cases:

1) Suppose that every point $m\neq q$ where a is normal to T_mM satisfies

$\det\langle\sigma(\cdot,\cdot),a\rangle_m>0$. Then, we have $\det d^2h_{a_m}>0$ and h_a is a Morse function.

We shall conclude that M^n is a homology sphere: The index of d^2h_{aq} is n. Since $\det d^2h_{a_m}>0$, the index of $d^2h_{a_m}$ is even. Then, with the notation of (2.2), $\tau_k=0$ and consequently, $\beta_k=0$ as soon as k is odd, $k\neq n$.

If we replace a by $-a$, we can conclude, in the same way, that $\beta_k=0$ if k is even, $k\neq 0$.

Consequently, all the Betti numbers of M^n are null, except β_0 and β_n. Thus M^n is a homology sphere.

2) Suppose that there exists two points q and q' such that

$$\det\langle\sigma(\cdot,\cdot),a\rangle_q=\det d(^2h_{a_q}<0),\quad \text{and}\quad \det\langle\sigma(\cdot,\cdot),a\rangle_{q'}=\det d^2h_{aq'}<0.$$

If m is a critical point of h_a, such that $m\neq q$, $m\neq q'$, we have, by assumption: $\det d^2h_{a_m}=\det\langle\sigma(\cdot,\cdot),a\rangle_m>0$.

Then h_a is a Morse function. Let s be the index of $d^2h_{aq'}\cdot s$ is odd.

We need now the following lemmas.

LEMMA 1. *Under the assumptions of* 2), $\beta_k=0$ *if* $k\neq 0, s, n-s, n$.

Proof. If k is odd, $k\neq n$, s, then $\tau_k=0$. Consequently, $\beta_k=0$ if $k\neq n$, s. Replacing a by $-a$, we conclude that $\beta_k=0$ if k is even, $k\neq 0, n-s$. Then, only $\beta_0, \beta_s, \beta_{n-s}, \beta_n$ are eventually not null.

LEMMA 2. *Under the assumptions of* 2), *the Stiefel-Whitney numbers of* M^n *are null.*

Proof. We have $H_k(M^n,\mathbf{Z}/2\mathbf{Z})=0$ if $k\neq 0$, s, $n-s$, n. Then,
$H^{n-k}(M^n,\mathbf{Z}/2\mathbf{Z})=0$ if $k\neq 0$, s, $n-s$, n. That is,
$H^k(M^n,\mathbf{Z}/2\mathbf{Z})=0$ if $k\neq 0$, s, $n-s$, n.

Consequently, the k-th-Stiefel-Whithney classes of M^n is null if $k\neq 0$, s, $n-s$, n.

On the other hand, if $\bar\omega_k$ denotes the k-th-inversed Stiefel-Whitney class of M^n, we have $\bar\omega_k=0$ as soon as $k>p$ (*cf.* [2]).

Suppose that $s<n-s$. We have

$$\begin{aligned}\bar\omega_n&=\omega_{n-1}\bar\omega_1+\omega_{n-2}\bar\omega_2+\cdots+\bar\omega_{n-s}\bar\omega_s+\cdots+\omega_s\bar\omega_{n-s}+\cdots+\omega_n\\&=\omega_{n-s}\bar\omega_s+\omega_s\bar\omega_{n-s}+\omega_n.\end{aligned}$$

Since $n>2p$, $\bar\omega_n=0$, and $n-s>p$. Then, $\bar\omega_{n-s}=0$, and

$$\omega_{n-s}\bar\omega_s+\omega_n=0. \tag{1}$$

We shall prove now that $\omega_{n-s}=0$, and $\omega_n=0$. We have:

$$\bar\omega_{n-s}=0=\omega_{n-s-1}\bar\omega_1+\omega_{n-s-2}\bar\omega_2+\cdots+\omega_s\bar\omega_{n-2s}+\cdots+\omega_{n-s}$$

with $\omega_{n-s-1}=\cdots=\omega_{s+1}\cdots 0$, and $\bar\omega_{n-2s}=0$ (for $n-2s\neq s$). Consequently $\omega_{n-s}=0$. Using (1), we conclude that $\omega_n=0$. Thus, only ω_s is eventually not null.

Consider now a Stiefel–Whitney number N,

$$N=\omega_1^{r_1}\omega_2^{r_2}\cdots\omega_n^{r_n}, \text{ with } 1r_1+\cdots+nr_n=n.$$

The only non null Stiefel–Withney number is eventually ω_s^l. In this case, n is a multiple of s, say $n=ls$, where l is odd, $l\neq 1$, $l\neq 3$. Since $\omega_s^2\in H^{2s}(M^n, \mathbf{Z}/2\mathbf{Z})=0$, $\omega_s^l=0$ and $N=0$. Thus, all the Stiefel–Whitney numbers of M^n are null.

The case where $n-s<s$ can be treated with the same method. We can now end the proof of the theorem applying the theorem of Thom (*cf.* par 2, 3).

References

1. J. Milnor, *Morse theory* (1963), Annals of Mathematics Studies, Princeton University Press.
2. J. Milnor and J.D. Stasheff, *Characteristic classes* (1974), Annals of Mathematics Studies, Princeton University Press.

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