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A Remark on the Krull Dimension

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1. Introduction

Lct A be a commutative local ring and $\mathcal{C}_A$ the Category of finite A—mudules. Then d: $\mathcal{C}_A \longrightarrow N$ defined by d(E), to be krull dimension of E satisfies the following properties:

(1) dim $(A/\underline{m})=0$

(2) If $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ is an exact sequence then $d(E)=\text{Max}(d(E'), d(E''))$.

(3) If $0 \longrightarrow aE \longrightarrow E \longrightarrow E/aE \longrightarrow 0$ where $a \in \underline{m}$ is an exact sequence then $d(E)=1+d(E/aE)$,

A main purpose of this note is to show that the above three properties do characterize dimension i.e., d: $\mathcal{C}_A \longrightarrow N$ with the above three properties is unique. For the sake of readers, we also give a proof of above propertics for krull dimension based on the notion of Hilbert-Samuel polynomial.

2. Definitions and Preliminaries.

Let A be a noetherian local ring with maximal ideal $\underline{m}$, E a finite A—module, (E_n) a stable $\underline{m}$—filteration of E.

Let $gr(A)=\bigoplus_{\nu=0}^{\infty}\underline{m}^{\nu}/\underline{m}^{\nu+1}$, $gr(E)=\bigoplus_{\nu=0}^{\infty}\underline{m}^{\nu}E/\underline{m}^{\nu+1}E$, $gr_{\nu}(A)=\underline{m}^{\nu}/\underline{m}^{\nu+1}$, then $gr_0(A)=A/\underline{m}$ is a field and hence $gr(A)$ is a noetherian ring, and $gr(E)$ is a finite $gr(A)$—module. $gr_{\nu}(E)=\underline{m}^{\nu}E/\underline{m}^{\nu+1}E$ is a noetherian A—module annhilated by $\underline{m}$.

If $\{x_1, x_2, \cdots\cdots, x_\rho\}$ generates $\underline{m}$, the image $\bar{x}_i$ of the $x_i \in \underline{m}$ generate $gr(A)$ as an $A/\underline{m}$—algebra and $\bar{x}_i$ has degree 1.

Proposition 1. *$P_E(t)=\sum_{\nu=0}^{\infty} C_\nu t^\nu$ where $C_\nu=[\underline{m}^{\nu}E/\underline{m}^{\nu+1}E : A/\underline{m}]$ is of the form $f(t)/(1-t)$ where $f(t) \in Z[t]$.*

Proof. We shall prove by the induction on ρ, the number of generators of $gr(A)$ over $A/\underline{m}$. Let $\rho=0$, then $\underline{m}^n/\underline{m}^{n+1}=0$ for all $n>0$, so that $gr(A)=A/\underline{m}$ and $gr(E)$ is a finitely generated $A/\underline{m}$—vector space, and hence $\underline{m}^nE/\underline{m}^{n+1}E=0$ for all $n>0$. Thus $P_E(t)$ is a polynomial.

Suppose $\rho>0$ and the proposition truc for $\rho-1$. Multiplication by $\bar{x}_\rho$ is an A—module homomorphism of $\underline{m}^n E/\underline{m}^{n+1}E$ into $\underline{m}^{n+1}E/\underline{m}^{n+2}E$ and hence it gives an exact sequence:

$$0 \text{——} P_n/P_{n+1} \longrightarrow \underline{m}^n E/\underline{m}^{n+1}E \longrightarrow \underline{m}^{n+1}E/\underline{m}^{n+2}E \longrightarrow Q_{n+1}/Q_{n+2} \longrightarrow 0 \cdots\cdots *1$$

Lct $P=\bigoplus_{\nu=0}^{\infty} P_\nu/P_{\nu+1}$, $Q=\bigoplus_{\nu=0}^{\infty} Q_{\nu+1}/Q_{\nu+2}$. These are both finitely generated A—modules and both annhilated by $\bar{x}_\rho$, hence they are $A/\underline{m}[\bar{x}_1, \bar{x}_2, \cdots\cdots, \bar{x}_\rho]$—module. Applying an additive function to *1 we get

$$\lambda(P_n/P_{n+1}) - \lambda(\underline{m}^n E/\underline{m}^{n+1}E) + \lambda(\underline{m}E/\underline{m}E) - \lambda(Q_{n+1}/Q_{n+2}) = 0.$$

Multiplying by t^{n+1} and summing with resepct to n we get $(1-t)P_E(t) = P_Q(t) - tP_P(t) + h(t)$ where $h(t)$ is a polynomial.

By the induction assumption $P_Q(t)$ and $P_P(t)$ are rational function of the form $g(t)/(1-t)$, and hence

$$(1-t)P_E(t) = P_Q(t) - tP_P(t) + h(t) = f(t)/(1-t)^\rho.$$

Therefore $P_E(t) = f(t)/(1-t)^{\rho+1}$.

Corollary. *For all $n>0$, $\lambda(\underline{m}^n E/\underline{m}^{n+1}E)$ is a polynomial in n of degree $\rho-1$.*

Proof. By the proposition $\lambda(\underline{m}^n E/\underline{m}^{n+1}E)$ is the coefficient of t^n in $f(t)/(1-t)^\rho$. Suppose $f(1)\neq 0$ and $f(t)=\sum_{k=0}^{N} a_k t^k$.

Since $(1-t)^{-\rho} = \sum_{k=0}^{\infty} \binom{\rho+k-1}{\rho-1} t^k$, $\lambda(\underline{m}^n E/\underline{m}^{n+1}E) = \sum_{k=0}^{N} a_k \binom{\rho+n+k-1}{\rho-1}$ for all $n \geq N$.

Therefore $\varphi\ \mathrm{gr}^{(n)}(E) = \lambda(\underline{m}^n E/\underline{m}^{n+1}E)$ is a polynomial in n of degree $\leq \rho-1$. It follows that the function

$$g_E(n) = \lambda_{\underline{m}}^E(n) = \lambda(E/\underline{m}^n E) = \sum_{i=0}^{n-1} \lambda(\underline{m}^i E/\underline{m}^{i+1}E)$$

is also a polynomial in n of degree $\leq \rho$ for all $n>0$.

This $g_E(n)$ is the Hilbert-Samuel polynomial in n of E with respect to $\underline{m}$. We shall let $\deg g_E(n) = d(E)$.

Remark Define $\delta(E)$ to be the order of the pole at 1 in $P_E(t)$ as we know easily $\delta(E) = d(E)-1$. In fact

$$\lambda(E/\underline{m}^{n+1}E) = \lambda(E/\underline{m}E) + (\underline{m}E/\underline{m}^2E) + \cdots\cdots + (\underline{m}^n E/\underline{m}^{n+1}E).$$

Put $P_E(t) = \sum_{\nu=0}^{\infty} C_\nu t^\nu$ where $C_\nu = \lambda(\underline{m}^\nu E/\underline{m}^{\nu+1}E)$ then since

$$C_\nu = \lambda_\nu - \lambda_{\nu-1} \text{ where } \lambda_\nu = C_1 + C_2 + \cdots\cdots + C_\nu$$

$$P_E(t) = \sum_{\nu=0}^{\infty} (\lambda(E/\underline{m}^{\nu+1}E) - \lambda(E/\underline{m}^\nu E))t^\nu$$

$$= \sum \lambda(E/\underline{m}^{\nu+1}E)t^\nu - (\sum (\lambda(E/\underline{m}^\nu E)t^{\nu-1})t = (1-t)\sum \lambda(E/\underline{m}^{\nu+1}E)t^\nu.$$

Therefore

$$\sum \lambda(E/\underline{m}^{\nu+1}E)t^\nu = \frac{1}{1-t} P_E(t) = \frac{1}{1-t} \sum \lambda(\underline{m}^\nu E/\underline{m}^{\nu+1})t^\nu.$$

As we expect, $\delta(E) = d(E)-1$.

Proposition 2. *Let A, $\underline{m}$, E be as in the proposition 1 and C_A the category of finite A-modules.*

For any objects E, E', $E'' \in C_A$, if $O \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow O$ is an exact sequence of *finite A-module, then* $d(E)=Max(d(E'), d(E''))$.

Proof. From given exact sequence we know

$$O\longrightarrow E'+\underline{m}^nE/\underline{m}^nE\longrightarrow E/\underline{m}^nE\longrightarrow E''/\underline{m}^nE''\longrightarrow O$$

is an exact sequence of finite A-module, and so $E/E'+\underline{m}^nE\simeq E''/\underline{m}^nE''$.

Since $\lambda(E''/\underline{m}^nE'')=\lambda(E/E'+\underline{m}^nE)\leq\lambda(E/\underline{m}^nE)$ we get $d(E'')\leq d(E)$ Furthermore,

$$\lambda_{\underline{m}}^{E}(n)-\lambda_{\underline{m}}^{E''}(n)=\lambda(E/\underline{m}^nE)-\lambda(E''/\underline{m}^nE'')$$
$$=\lambda(E/\underline{m}^nE)-\lambda(E/E'+\underline{m}^nE)=\lambda(E'+\underline{m}^nE/\underline{m}^nE)=\lambda(E'/\underline{m}'\cap\underline{m}^nE)$$

and there exists $r>0$ such that $E'\cap\underline{m}^nE\subseteq\underline{m}^{n-r}E'$ for all $n>r$ by Artin-Rees. Thus

$$\lambda(E'/\underline{m}^nE')\geq\lambda(E'/E'\cap\underline{m}^nE)\geq\lambda(E'/\underline{m}^{n-r}E').$$

This means that

$$\lambda_{\underline{m}}^{E}(n)-\lambda_{\underline{m}}^{E''}(n) \text{ and } \lambda_{\underline{m}}^{E'}(n)$$

have the same degree and the same leading term.

Proposition 3. *Let A,* $\underline{m}$, *E be as in proposition* 1, *and* $a\in\underline{m}$ *non-zero diviser on E. Then* $d(E)-1=(E/aE)$

Proof. From the given condition, we get an exact sequence:

$$O\longrightarrow aE\longrightarrow E\longrightarrow E/aE\longrightarrow O$$

and hence

$$O\longrightarrow aE+\underline{m}^nE/\underline{m}^nE\longrightarrow E/\underline{m}^nE\longrightarrow E/aE/\underline{m}^n(E/aE)\longrightarrow O$$

and then

$$\lambda_{\underline{m}}^{E/aE}(n)=\lambda(E/aE+\underline{m}^nE)=\lambda(E/\underline{m}^nE)-\lambda(aE+\underline{m}^nE/\underline{m}^nE),$$
$$aE+\underline{m}^nE/\underline{m}^n\cong aE/aE\cap\underline{m}^nE\cong E/(\underline{m}^nE:a) \text{ and } \underline{m}^{n-1}E\subseteq(\underline{m}^nE:a).$$

Hence

$$\lambda_{\underline{m}}^{E/aE}(n)\geq(E/\underline{m}^nE)-\lambda(E/\underline{m}^{n-1}E)=\lambda_{\underline{m}}^{E}(n)-\lambda_{\underline{m}}^{E}(n-1).$$

It follows that $d(E/aE)\geq d(E)-1$. On the other hand, $aE\cong E$ as A-modules by the hypothesis on a. We have a_n exact sequence:

$$O\longrightarrow aE/aE\cap\underline{m}^nE\longrightarrow E/\underline{m}^nE\longrightarrow E/aE/\underline{m}^n(E/aE)\longrightarrow O.$$

Hence

$$\lambda(aE/aE\cap\underline{m}^nE)-\lambda(E/\underline{m}^nE)+\lambda(E/aE/\underline{m}^n(E/aE))=O$$

for all $n>0$.

By the Artin-Rees $aE\cap\underline{m}^nE$ is a stable $\underline{m}$-filteration of E.

Since $aE\cong E$, $\lambda(E/aE\cap\underline{m}^nE)$ and $\lambda_{\underline{m}}^{E}(n)$ have the same leading term because the degree and leading coefficient of Hilbert-Samuel polynomial depend only on E and $\underline{m}$, not on the filteration chosen. Therefore $d(E/aE)\leq d(E)-1$.

3. Main Theorem

Theorem: *Let A be a local noetherian ring with maximal ideal m, E finite A-module, gr* $(E)=\bigoplus_{\nu=0}^{\infty}\underline{m}^{\nu}E/\underline{m}^{\nu+1}E$, $P_E(t)=\sum_{\nu=0}^{\infty}C_{\nu}t^{\nu}$ where $C_{\nu}=[\underline{m}^{\nu}E/\underline{m}^{\nu+1}E:A/\underline{m}]$, $\delta(E)$ *the order of the pole at* 1 *in* $P_E(t)$ *and* C_A *the category of finite A-modules.*

Define $E\longrightarrow\lambda(E)\in N$ *non-negative, then followings hold:*

(1) $\underline{m}^nE=O$ *for some* $n>0\Longleftrightarrow$(1′) $\delta(E)=\delta(A/\underline{m})=O$

(2) $O \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow O$ *is exact* $\delta(E)=Max(\delta(E'), \ \delta(E''))$

(3) $O \longrightarrow E \longrightarrow E \longrightarrow E/aE \longrightarrow O$ *is eyact where*

$a \in \underline{m} \Longrightarrow (E/aE)=\delta(E)-1$.

Conversely this map is uniquely determined by the above conditions.

Proof. (1) $m^n E=O$ for some n implies $\underline{m}(\underline{m}^n E)=O$ and hence $\underline{m}^{n+1}E=O$. Since $\underline{m}^{n+k}E=O$ for all $k \geq 0$ $\underline{m}^{n+k}E/\underline{m}^{n+k+1}E=O$.

Thus $C_{n+k}=dim_{A/m}(\underline{m}^{n+k}E/\underline{m}^{n+k+1})=O$. Therefore $P_E(t)=\sum_{\nu=0}^{\infty} C_\nu t^\nu=\sum_{\nu=0}^{n} C_\nu t^\nu$ because $C_\nu=O$ if $\nu>n$ so the order of the pole is zero i.e., $\delta(E)=O$.

(1)$\Longleftrightarrow$(1′) From the given condition, we get a chain

$$E \supset E_1 \supset \cdots\cdots \supset E_r=O$$

of submodules such that $E_i/E_{i+1} \simeq A/P_i$ by J. P. Serre.

Since $P_i \supseteq \underline{m} \Longrightarrow P_i \supseteq \underline{m}$, $P_i=\underline{m}$. Hence $\delta(E)=\delta(A/P)=O$.

(2) & (3) followes from proposition 2, 3 and remark. conversely, by J. P. Serre, there exists a chain such that $E=E_0 \supset E_1 \supset \cdots\cdots \supset E_r=O$ such that $E_i/E_{i+1} \simeq A/P_i$ for some i.

From $\quad O \longrightarrow E_1 \longrightarrow E \longrightarrow E/E_1 \longrightarrow O$

$\quad O \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_1/E_2 \longrightarrow O$

$\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

We know $d(E)=Max(d(E/E_1), \ d(E_1/E_2), \cdots\cdots, d(E_{r-1}/E_r)$

$=Max(d(A/P_1), \ d(A/P_2), \cdots\cdots, d(A/P_r))$.

If $d(A/P)=\dim(A/P)$, then $d(E)=\dim(E)$.

Thus if $d(E) \neq \dim(E)$ for some module E then $d(A/P) \neq \dim(A/P)$ for some prime ideal P.

Assume that $d(E) \neq \dim(E)$, and then choose a maximal one among all pime ideals P for which $d(A/P) \neq \dim(A/P)$.

Let P be a maximal one.

(a) $P \neq \underline{m}$ because if $P=\underline{m}$ then $d(A/P)=0$ by (1′) whereas $\dim(A/\underline{m})=0$.

(b) Since $P \subsetneq \underline{m}$ we can pick $a \in \underline{m}-P$. The multiplication by a in A/P is one-one i.e., $O \longrightarrow A/P \longrightarrow A/P \longrightarrow A/P+aA \longrightarrow O$ is exact.

Then by (3) $d(A/P+aA)=d(A/P)-1$ i.e., $d(A/P)=1+d(A/P+aA)$.

However choose $A/P+aA=E \supset E_1 \supset \cdots\cdots \supset E_r=O$ such that $E_i/E_{i+1} \cong A/P_i$, then $P_i \supset P+aA \supsetneq P$.

Because P was a maximal amongest $d(A/P) \neq \dim(AP)$ we must have $d(A/P_i)=\dim(A/P_i)$ for all i. Therefore

$$Max(d(E/E_1), \ d(E_1/E_2), \cdots\cdots\cdots)=d(A/P+aA)$$
$$=Max(d(A/P_1), \ d(A/P_2), \ d(A/P_3), \cdots\cdots)=\dim(A/P+aA).$$

Hence

$$d(A/P)=1+d(A/P+aA)=1+\dim(A/P+aA)=\dim(A/P),$$

which is a contradiction.

Corollary *Let* $A \longrightarrow B$ *be a local map of noetherian local rings, and E a finite B-module which is A-flat. Then for any finite A-module M we have* $dim_A(M)=dim_B(M_A \otimes E)-dim_B(E/\underline{m}E)$.

Proof. Let $C_A \longrightarrow N$ where $\delta(M) = \dim_B(M_A \otimes E) - \dim_B(E/\underline{m}E)$.

(i) $\delta(A/\underline{m}) = \dim_B(A\underline{m} \otimes E) - \dim_B(E/\underline{m}E) = \dim_B(E/\underline{m}E) - \dim_B(E/\underline{m}E) = O$

(ii) $O \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow O$ is exact

$O \longrightarrow M'_A \otimes E \longrightarrow M_A \otimes E \longrightarrow M''_A \otimes E \longrightarrow O$ is exact since E is A-flat.

So $\dim_B(M_A \otimes E) = \mathrm{Max}(\dim_B(M'_A \otimes E),\ \dim_B(M''_A \otimes E))$.

Therefore

$$\delta(M) = \mathrm{Max}(\delta(M'),\ \delta(M'')).$$

(iii) $O \longrightarrow M \longrightarrow M \longrightarrow M/aM \longrightarrow O$ where $a \in \underline{m}$ is exact

$$\Longrightarrow O \longrightarrow M_A \otimes E \longrightarrow M_A \otimes E \longrightarrow M/aM_A \otimes E \longrightarrow O$$

$$\Longrightarrow O \longrightarrow M_A \otimes E \longrightarrow M_A \otimes E \longrightarrow M_A \otimes E/a(M_A \otimes E) \longrightarrow O$$

$$\Longrightarrow \dim_B(M_A \otimes E) = 1 + \dim_B(M \otimes E/a(M \otimes E)).$$

So $\delta(M) = 1 + (M/aM)$.

By uniqueneess

$$\dim_A(M) = \dim_B(M_A \otimes E) - \dim_B(E/\underline{m}E).$$

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