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A note on topological entropy of diffeomorphisms

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1. Introduction

The purpose of this note is to give some estimations for topological entropy of diffeomorphisms which preserve foliations.

Throughout this note, M denotes a closed m-dimensional C^∞-manifold, E an orientable p-dimensionel C^∞-manifold of M and f a diffeomorphism of M which preserves E. Fix a Riemann metric on M. Let d and d_L denote the induced distance functions on M and on a leaf L of E, respectively.

Hereafter $B_r(x)$ denotes $\{y\in M;\ d(x,y)\leq r\}$ and similarly $D_r(x)$ denotes $\{y\in L;\ d_L(x,y)\leq r\}$ where L is the leaf which contains x.

Also fix a family of distinguished charts $\{D_i^p\times D_i^{m-p}\}_{i=1,\cdots,N}$ whose interiors cover M and an associated partition of unity $\{\lambda_i\}$.

We put $P_z=D_i^p\times\{Z\}$ and call this set a plague. If necessary, taking a refinement, we may assume that for every plague P_z and for every index j, the set $P_z\cap D_j^p\times D_j^{m-p}$ is contained in some plague $P_{z'}$, *for* $z'\in D_j^{m-p}$.

Definition. We say that a subset X of M (n,δ)-*spans* M with respect to f if for any $x\in M$ there exists an $x'\in X$ such that $d(f^i(x),\ f^i(x'))<\delta$ for $i=0,\cdots,n-1$. We put $s_f(n,s)$ to be the minimum of the cardinalities of such subsets.

Then the *topological entropy of* f is defined by the following:

$$h(f)=\sup_{\delta>0}\ \lim\ \sup_n \frac{1}{n}\log(S_f(n,\delta))$$

Next, we define the volume expanding ratio of f,

$$\rho(f)=\lim_n\ sup\ \frac{1}{n}\log(\sup_{x\in M} volume(f^n(D_r(x))))$$

Let T_p denote the typological linear space consisting of p-dimensional smooth forms on M are T'_p the dual of T_p.

Define the closed convex cone C of T'_p to be one that is generated by all Dirac currents at positive p-vectors tangent to E and call an element of C a foliation cycle, if it is closed as a current.

Here positive means the compatibility to some fixed orientation of E. We also call a closed current a foliation cycle if it is represented as a difference of foliation cycles.

2. Main theorem.

Lemma 1. *For every foliation cycle α there exists a family signed measure μ_i on D_i^{m-p} $i=1, \cdots, N$ such that*

$$\langle \alpha, \eta \rangle = \sum_i \int_{D_i^{m-p}} \left(\int_{P_z} \lambda_i \eta \right) d\mu_i(z),$$

where η denotes an arbitrary p-form on M.

Making use of this lemma, we define the volume of a foliation cycle as follows.

Let α be a foliation cycle and μ_i be as in Lemma 1. By Haln's theorem, each μ_i has the unipue decomposition as $\mu_i = \mu_{i+} - \mu_{i-}$ where μ_{i+} and μ_{i-} are non-negative measures and each of them is singular with respect to the other. Let $\bar{\mu}_i = \mu_{i+} + \mu_{i-}$.

Definition. Volume $(\alpha) = \sum_i \int_{D_i^{m-p}} \left| \int_{P_z} \lambda_i v \right| d\bar{\mu}_i(z)$

Here v denotes the volume form on each plague induced by the Riemann metric.

The following Lemma is immediate by the definition.

Lemma 2. *For any foliation cycle α, we have*

$$\rho(f) \geq \lim_n sup \frac{1}{n} \log(volume\ ((f_*)^n \alpha))$$

Let $h(f)$ denote the topological entropy of f and $\delta(f)$ the volume expanding ratio of f.

Theorem. $h(f) \geq \delta(f)$

proof. Let $\hat{P}(x) = \bigcup_{x \in P_z} P_z$ and $\hat{B}_r(x) = B_r(x) \cap \hat{P}(x)$.

If one takes δ sufficiently small, then, for every $x \in M$,

$$f(\hat{B}_\delta(x)) \cap B_\delta(f(x)) \subset \hat{B}_\delta(f(x)).$$

If $\rho(f) \leq 0$ there is nothing to prove.

Otherwise, it suffices to show that for every positive. $\rho < \rho(f)$, $h(f) \geq \rho$. Fix a positive $\rho < \rho(f)$ and also fix r small enough that for every $x \in M$, $D_{2r}(x) \subset \hat{B}_\delta(x)$.

Then, if $y \in D_r(x)$ we have that $D_r(x) \subset D_{2r}(y) \subset \hat{B}_\delta(y)$. For every positive integer n_0 there exist an integer $n \geq n.$ and $x \in M$ satisfying volume $(f^n(D_r(x))) \geq e^{n\rho}$ Let $D = D_r(x)$ and s be $s_f\left(n+1, \frac{\delta}{2}\right)$.

Take a subset $\{x_1, \cdots, x_s\}$ of M which $\left(n+1, \frac{\delta}{2}\right)^-$ spans M and put $X_j = \{y \in D;$ $d(f^i(x_j), f^i(y)) < \frac{\delta}{2}\ \ i=0, \cdots, n\}$.

We may assume that $X_1 \neq \phi, \cdots, X_s \neq \phi$ and $X_{s+1} = \cdots = X_s = \phi$. Take $y_j \in X_j$ for $j=1, \cdots, s'$.

Then we have $f^k(X_j) \subset B_\delta(f^k(y_j))$ for $k=0, \cdots, n$.

Henceforce, by the fact $X_j \subset D \subset \hat{B}_\delta(y_j)$, we get that $f^n(X_j) \subset \hat{B}_\delta(f^n(y_j))$.

Let K be an upper bound for the volumes of plagues.

Then, for every $x \in M$, volume $(\hat{B}_\delta(x)) \leq$ volume $(\hat{P}(x)) \leq NK$.

Therefore $s_f\left(n+1, \frac{\delta}{2}\right) \geq s' \geq \frac{1}{NK} e^{n\rho}$.

Since we can take n arbitrarily large,

$$h(f) \geq \lim_{n} sup \frac{1}{n+1} \log\left(s_{\rho}(n+1, \frac{\delta}{2})\right) \geq \rho$$

This completes the proof of theorem.

Finally we remark that theorem holds also for continuous maps from M to itself which preserves E.

References

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