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On Lindelöf Degree and Compactness

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The purpose of this note is to characterize the compactness of a topological space by the Lindeloef degree of the topological space.

If X is a topological space and χ_u is the least cardinal number such that every open cover of X has a subcover of cardinality less than or equal to χ_u, then χ_u is said to be the Lindeloef degree of the topological space X. Using this concept we prove the following theorems.

Theorem 1. *A topological space is compact if and only if every net whose indexing set has cardinality less than or equal to the Lindeloef degree has a convergent subnet or equivatently a cluster point.*

Proof. The proof of the necessity is well known. To prove the sufficiency let us assume that X is a topological space in which every net whose indexing set with cardinality less than or equal to the Lindeloef degree has a cluster point. Let $\mathcal{F}$ be an open covering of X. Then there is a subcover $\mathcal{G}$ with the cardinality less than or equal to the Lindeloef degree. For each finite subset $\mathcal{W}$ of $\mathcal{G}$, let $A_W = \cap \{X-W : W \in \mathcal{W}\}$ and let $\mathcal{A} = \{A_W : \mathcal{W}$ is a finite subset of $\mathcal{G}\}$. Then each element of $\mathcal{A}$ is closed and nonempty unless $\mathcal{F}$ has a finite subcober. Clearly the cardinality of $\mathcal{A}$ is less than or equal to the Lindeloef degree. $\mathcal{A}$ has the finite intersection property. $\mathcal{A}$ can be directed by setting $A < B$ if and only if $A \supset B$. For each A in $\mathcal{A}$, we choose s_A in A so $\{s_A : A \in \mathcal{A}\}$ is a net. Let s be a cluster point of the net $\{s_A : A \in \mathcal{A}\}$.

If $A < B$, then $s_B \in A$ so that the net is eventually in every member of $\mathcal{A}$. Since these members of $\mathcal{A}$ are closed, s belongs to each member of $\mathcal{A}$ and so $\mathcal{G}$ is not a cover of X. This contradiction shows that $\mathcal{F}$ has a finite subcover and completes the proof of the theorem.

Theorem 2. *A topological space X is compact if and only if every infinite subset of X with cardinality lessthan or equal to the Lindeloef degree of X possesses a condensation point.*

Proof. Clearly the Condition is necessary. Assume that X is a topological space such that every infinite subset of X with cardinality less than or equal to the Lindeloef degree of X possesses a condensation point. Let $\mathcal{F}$ be an open covering of X and among all the subcovers of $\mathcal{F}$ choose a $\mathcal{V}$ with the minimal cardinality. Let χ_u be the cardinality of $\mathcal{V}$. Well order $\mathcal{V}$ by ω_{χ_u}. The set $X-V_0$ is nonempty. Therefore there exists some $x_{\beta_0} \in X-V_1$ such that β_0 is the least ordinal number such that $x_{\beta_0} \in V_{\beta_0}$. Since $\beta_0 < \omega_{\chi_u}$ the set $\{V_\alpha : \alpha \leq \beta_1\}$ is not a cover for X. Therefore there exists an $x_{\beta_1} \in X - \bigcup_{\alpha \leq \beta_0} V_\alpha$, where β_1 is the minimal ordinal such that $x_{\beta_1} \in V_{\beta_1}$.

Suppose that for every ordinal $\alpha < \gamma < \omega_{\alpha_u}$ we have an x_{β_α} such that if $\alpha' < \alpha < \gamma$ then $\beta_{\alpha'} < \beta_\alpha$ and $x_{\beta_{\alpha'}}$ is different from x_{β_α}. Let δ be $\sup_{\alpha<\gamma} \beta_\alpha$. Since ω_{α_u} is regular we have $\delta < \omega_{\chi_u}$. Now $X - \bigcup_{\alpha \leq \delta} V_\alpha$ is nonempty, therefore there exists an $x_{\beta_\gamma} \in X - \bigcup_{\alpha \leq \delta} V_\alpha$ where β_γ is the least ordinal such that $x_{\beta_\alpha} \in V_{\beta_\gamma}$.

Thus for every $\alpha < \omega_{\chi_u}$ we have a β_α and an x_{β_α}. Since the set of the x_{β_α} has cardinality χ_u there exists some x in X which is a condensation point of this set. Since $\mathcal{V}$ is a cover for X, there exists a $\beta < \omega_{\chi_u}$ such that $x \in V_\beta \in \mathcal{V}$. Since V_β is a neighborhood of x, V_β contains χ_v members of the x_{β_α}, but this is a contradiction for if $\beta_\alpha > \beta$ then x_{β_α} dose not belong to V_β. This completes the proof of the theorem.

It is easy to see the following

Corollary. *If X is a topological space and χ_u is the least cardinal number such that every subset A of X with cardinality greater than or equal to χ_u possesses a condensation point, then the Lindeloef degree of X is less than or equal to χ_u.*

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