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On M-spaces

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1. Introduction

Let m be a cardinal number$\geqq 1$. We will say that a topological space X is a $P(m)$-*space* if for a set Ω of power m and for any family $\{G(\alpha_1, \cdots, \alpha_n) \mid \alpha_1, \cdots, \alpha_n \in \Omega,\ n=1,2,\cdots\}$ of open subsets of X such that $G(\alpha_1, \cdots, \alpha_n) \subset G(\alpha_1, \cdots, \alpha_n, \alpha_{n+1})$ for $\alpha_1, \cdots, \alpha_n, \alpha_{n+1},\ n=1,2,\cdots$, there exists a family $\{F(\alpha_1, \cdots, \alpha_n) \mid \alpha_1, \cdots, \alpha_n \in \Omega,\ n=1,2,\cdots\}$ of closed subsets of X satisfying the following conditions:

(1) $F(\alpha_1, \cdots, \alpha_n) \subset G(\alpha_1, \cdots, \alpha_n)$ for $\alpha_1, \cdots, \alpha_n \in \Omega,\ n=1,2,\cdots$

(2) $X = \bigcup_{n=1}^{\infty} F(\alpha_1, \cdots, \alpha_n)$ for any sequence $\{\alpha_n\}$ such that $X = \bigcup_{n=1}^{\infty} G(\alpha_1, \cdots, \alpha_n)$.

In case X is a $P(m)$-space for any cardinal number m, we will say that X is a P-space.

We will say that a topological space X is an *M-space* if there exists a normal sequence $\{U_n \mid n=1,2,\cdots\}$ of open coverings of X satisfying the following condition (3):

(3) If a family R consisting of a countable number of subsets of X has the finite intersection property and contains as a member a subset of $St(x_0, U_n)$ for every n and for some fixed point x_0 of X, then $\cap \{K \mid K \in R\} \neq \phi$.

This note is a study of M-spaces.

2. M-spaces.

It is easy to see that metrizable spaces and countably compact spaces are M-spaces.

K. Morita has shown the following theorem [1].

Theorem 1. *A topological space X is an M-space if and only if there exists a closed continuous mapping f from X onto a metrizable space T such that $f^{-1}(t)$ is countably compact for each point of T.*

Recently Z. Frolik [2] has shown that a completely regular Hausdorff space X is paracompact and topologically complete in the sense of Čech if and only if there exists a closed continuous map f from X onto a complete metric space T such that $f^{-1}(t)$ is compact from each point t of T. Our Theorem 1 above gives the essential part of

Frolik's theorem by Theorem 2.

Theorem 2. *A paracompact normal space which is G_δ in a countably compact regular space is an M-space.*

Proof. Let R be a countably compact regular space and X a subspace of R such that X is paracompact and normal and X is G_δ in R. Then there exist a countable number of open subsets G_n, $n=1, 2, \cdots$ of R such that $X=\cap G_n$. Since X is paracompact normal and R is regular, there exists a normal sequence $\{U_n | n=1, 2, \cdots\}$ of locally finite open coverings of X such that the closure of each set of U_n in R is contained in G_n. Then it is easy to see that $\{U_n | n=1, 2, \cdots\}$ satisfies condition (3). Thus X is an M-space.

However, there exists an M-space which is not even normal; for example, the topological product of a countably compact, non-normal space with a normal metric space is such a case.

Since a metrizable space is a P-space, we obtain the following theorem.

Theorem 3. *An M-space is a P-space.*

Proof. Let $\{G(\alpha_1, \cdots, \alpha_n) | \alpha_1, \cdots \alpha_n \in \Omega,\ n=1, 2, \cdots\}$ be a family of open subsets of X such that $G(\alpha_1, \cdots, \alpha_n) \subset G(\alpha_1, \cdots, \alpha_n, \alpha_{n+1})$, for $\alpha_1, \cdots, \alpha_{n+1} \in \Omega$ where Ω is a set of indices. Let X be an M-space: let $\{U_n | n=1, 2, \cdots\}$ be a normal sequence of open coverings of X satisfying condition (3). Let us put $F(\alpha_1, \cdots, \alpha_n) = X - St(X - G(\alpha_1, \cdots, \alpha_n), U_n)$. Then we have clearly $F(\alpha_1, \cdots, \alpha_n) \subset G(\alpha_1, \cdots, \alpha_n)$. If $X = \bigcup_{n=1}^{\infty} G(\alpha_1, \cdots, \alpha_n)$, then $X = \bigcup_{n=1}^{\infty} F(\alpha_1, \cdots, \alpha_n)$; otherwise there will exist a point $x_0 \in X - \bigcup_{n=1}^{\infty} F(\alpha_1, \cdots, \alpha_n) = \bigcap_{n=1}^{\infty} St(X - G(\alpha_1, \cdots, \alpha_n), U_n)$, and hence $[X - G(\alpha_1, \cdots, \alpha_n)] \cap St(x_0, U_n) \neq \phi$ for $n=1, 2, \cdots$, and consequently $\bigcap_{n=1}^{\infty} [X - G(\alpha_1, \cdots, \alpha_n)] \neq \phi$ by (3), but this is a contradiction.

References

[1.] K. Morita : Products of normal spaces with metric spaces. *Math. Annalen* 154, 365-382 (1964).

[2.] Z. Frolik : On the topological product of paracompact spaces. *Bull. acad. polon. sci.* 8, 747-750 (1960).

[3.] C. H. Dowker : On countably paracompact spaces. *Canad. J. Math.* 1, 219-224 (1951).