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A Note on C-Semistratifiable (Mod K) Spaces

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It was shown that a regular space is metrizable if and only if it is c-semistratifible wM space [2]. In this paper we will show that a regular space is metrizable if and only if it is c-semistratifiable (mod K) wM-space. All spaces used here are T_2-space and all undefined terms and notations may be found in [6].

A c-semistratification for a topological space X is a system $\{g(n, x) \mid x \in X, \ n=1, 2, \cdots\}$ of open subsets of X which satisfies the following conditions;

(1) $x \in g(n, x)$

(2) $g(n+1, \ x) \subset g(n, x)$

(3) If A is a closed compact subset of X and $x \in X-A$, then there exists n such that $x \notin g(n, a)$ for each $a \in A$.

A space X is said to be c-semistratifiable if X has a c-semistratification.

Let $(X, \mathcal{T})$ be a topological space and let g be a function $g : N \times X \rightarrow \mathcal{T}$, then g is called a COC-function for X if it satisfies these two conditions;

(1) $x \in \bigcap_{n=1}^{\infty} g(n, x)$ for all $x \in X$

(2) $g(n+1, \ x) \subset g(n, x)$ for all $n \in N$ and $x \in X$.

A topological space X is c-semistratifiable (mod K) space if there exists a compact covering $\mathcal{K}$ of X and COC-function g for X such that; if $x \in K \in \mathcal{K}$, $K \subset U$ for a cocompact open set U in X and $\{x_n\}$ is a sequence of points in X with $x \in g(n, x_n)$ for all n, then $\{x_n\}$ is eventually in U.

Theorem 1. *If X is a c-semistratifiable, then X is a c-semistratifiable (mod K).*

Proof Since X is a c-semistratifiable, there exists a COC-function such that if $x \in g(n, x_n)$ for all $n \in N$ and $x \in U$ for a cocompact open set U, then $\{x_n\}$ is eventually in U. Hence if $x \in K \in \mathcal{K}$ for a compact covering $\mathcal{K}$ of X, $K \subset U$ and $\{x_n\}$ is a sequence of points in X with $x \in g(n, x_n)$ for all n, then $\{x_n\}$ is clearly eventually in U.

Let $\mathcal{U}_1, \mathcal{U}_2, \cdots$ be a sequence of covers of X. If x and y are distinct points of X, then there exists $n \in N$ such that $y \notin St^k(x, \mathcal{U}_n)$. A space with a sequence of open covers satisfying above conditions is said to have $\bar{G}_\delta(k)$-diagonal and a space with a σ-closure preserving separating closed cover is called a $\sigma^{\#}$-space. The followings are clear from the results in [2].

Theorem 2. *Any space with a $\bar{G}_\delta(1)$-diagonal is c-semistratifiable (mod X).*

Theorem 3. *Every $\sigma^{\#}$-space is c-semistratifiable (mod K).*

A topological space X is β-space provided that there is a COC-fuction that if $x \in g(n, x_n)$ for $n=1, 2, \cdots$, then $\{x_n\}$ has a cluster point. It was proved that a space is semistratifiable if and only if it is both a β-space and a $\sigma^{\#}$-space [1] or a c-semistratifiable β-space [2]. But it remains true when c-semistratifiability is replaced by c-semistratifiable (mod K).

Theorem 4. *A regular space X is semistratifiable if and only if it is a c-semistratifiable (mod K) β-space.*

Proof A regular semistratifiable space is clearly a c-semistratifiable (mod K) β-space. For the converse, let X be a regular c-semistratifiable (mod K) space, then there is a COC-function g with $\text{cl}g(n+1, x) \subset g(n, x)$ for all $x \in X$ and $n \in N$ such that if $x \in K \in \mathcal{K}$ for a compact covering $\mathcal{K}$ of X, $K \subset U$ and $\{x_n\}$ is a sequence of points in X with $x \in g(n, x_n)$ for all n, then $\{x_n\}$ is eventually in U. Let $x, x_n \in X$ such that $x \in g(n, x_n)$ for $n=1, 2, \cdots$. We will show that $\{x_n\}$ converges to x. The sequence $\{x_n\}$ has at least one cluster point x, moreover, every subsequence of $\{x_n\}$ also has at least one cluster point. Suppose y is a cluster point of $\{x_n\}$ distinct from x. Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $x_{n_i} \in g(i, y)$ for $i=1, 2, \cdots$ and $x_{n_i} \neq x$ for all i. Since $\text{cl}g(i+1, y) \subset g(i, y)$, y is the only one cluster point of $\{x_{n_i}\}$. It follows that $x_{n_i} \to y$, so that $C = \{y\} \cup \{x_{n_i} \mid i=1, 2, \cdots\}$ is compact. Since X is regular, there exists m such that $x \notin g(m, c)$. Take $k > m$, then $x \notin g(m, x_k) \supset g(k, x_k)$, which is a contradiction. Hence x is the unique cluster point of $\{x_n\}$. Since every subsequence of $\{x_n\}$ has a cluster point, $\{x_n\}$ converges to x. This completes the proof.

Let $(X, \mathcal{T})$ be a space and let g be a fuction from $N \times N$ into $\mathcal{T}$ such that $x \in \bigcap_{n=1}^{\infty} g(n, x)$ for each $x \in X$. The space X is a wΔ-space if $\{p, x_n\} \subset g(n, y_n)$ for $n=1, 2, \cdots$, then the sequence $\{x_n\}$ has a cluster point. Also X is a wM-space if $p \in g(n, z_n)$, $g(n, y_n) \cap g(n, z_n) \neq \phi$, and $x_n \in g(n, y_n)$ for$n=1, 2, \cdots$, then $\{x_n\}$ has a cluster point.

Corollary 5. *A regular space is developable if and only if it is a c-semistratifiable (mod K) wΔ-space.*

Proof Since any wΔ-space is a β-space, it is clear from Theorem 4 and [3].

Corollary 6. *A regular space is metrizable if and only if it is a c-semistratifiable (mod K) wM-space.*

Proof Note that every wM-space is a β-space and apply Theorem 4 and [2].

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