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**<Original>**

# Dynamic Stability of Anisotropic Cylindrical Shells

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# 異方性 圓筒쉘의 動的 安定性

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**抄 錄**

쉘이 複合材로 제작되었거나 補强材를 사용하였을 때에는 쉘要素의 力學的 特性은 方向性을 가지게 된다. 이 論文에서는 線型關係式을 사용하여 動的 表面荷重을 받는 異方性圓筒쉘의 安全性을 고려하여 다루었다. 材料의 異方性을 고려하여 3方向 즉, 軸방향, 둘레방향 및 반경방향의 慣性項이 포함된 運動方程式을 유도 하였다. 動的 安定性의 解析은 Bolotin의 方法에 따랐으며 近似解를 行列式型으로 제시하였다. 數值例로 外壓을 받는 圓筒쉘의 挫丹을 다루었으며 本 理論과 古典理論 사이의 差異點을 檢討하였다.

## 1. Introduction

Composite materials such as boron-epoxy, glass-epoxy, reinforced plastics, and whiskers are increasingly used in advanced structural applications.(1) When a shell is made of composite material or reinforced with stiffening elements, its mechanical properties are usually directional.

General theory of anisotropic shells was developed by Ambartsumyan(2), and Dong et al.(3) The static stabilities of anisotropic shells have been studied extensively(4~8). There are few investigations on the dynamic stability of anisotropic cylindrical shells. Chen and Bert(9) deal with the dynamic stability of a thin-walled, circular cylindrical shell made of either composite or isotropic material and conveying an inviscid, incompressible liquid in helical swirling flow.

In this paper, the anisotropic cylindrical shells are treated with linearlized relations and the effects of end conditions are not considered. The cylindrical shell is subjected to dynamic surface loading. The inertia effects in three directions, which are along the axis, circumference, and transverse, are included in the present study of dynamic stability. The equations of motion are formulated and an approximate solution is given in determinant form.

## 2. Equations of Motion

Let $(u, v)$ represent the displacement components in the direction of the coordinate axes $(x, y)$ in Fig. 1. The notation $w$ is for the outward normal displacement. Also, the stress and strain components are denoted by $(\sigma_x, \sigma_y, \tau_{xy})$

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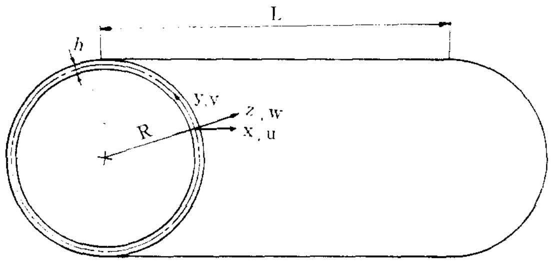


**Fig. 1**

and $(e_x, e_y, \gamma_{xy})$, respectively. For small deflection, the strain components on the middle surface are given

$$e_x = \partial u/\partial x, e_y = \partial v/\partial y + w/R,$$
$$\gamma_{xy} = \partial v/\partial x + \partial u/\partial y \qquad (1)$$

Based on the Kirchhoff-Love hypothesis, the stress-strain relations of an anisotropic material become[10]

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{16}\\ b_{12} & b_{22} & b_{26}\\ b_{16} & b_{26} & b_{66}\end{bmatrix}\begin{bmatrix}e_x\\ e_y\\ \gamma_{xy}\end{bmatrix} - \overline{T}\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_6\end{bmatrix} \qquad (2)$$

where $b_{ij}(i, j = 1, 2, \text{ or } 6)$ are anisotropic elastic constants, $\lambda_1$ is thermal expansion coefficients, and $\overline{T}$ is the temperature gradient.

The dynamic stability of the cylindrical shell will be studied by assuming certain small deviations from the position of undisturbed equilibrium and investigating changes in the disturbances with time[11]. Considering that both the undisturbed and disturbed states satisfy the equation of motion independently, we obtain the variational equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + X - \rho h\frac{\partial^2 u}{\partial t^2} = 0$$
$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + Y - \rho h\frac{\partial^2 v}{\partial t^2} = 0 \qquad (3)$$
$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{\partial^2 M_y}{\partial y^2} - \frac{N_y}{R} + Z - \rho h\frac{\partial^2 w}{\partial t^2} = 0$$

where $\rho$ and $t$ are density of the material and time, respectively. The notations $X$, $Y$, and $Z$ in the above equations are additional load components due to the normal pressure $q$ and are given as follows[12]:

$$X = qR\left(\frac{\partial^2 v}{\partial x\partial y} + \frac{1}{R}\frac{\partial w}{\partial x}\right)$$
$$Y = -qR\frac{\partial^2 u}{\partial x\partial y} \qquad (4)$$
$$Z = qR\left(\frac{\partial^2 w}{\partial y^2} + \frac{w}{R^2}\right)$$

The linear shell constitutive relations for an anisotropic cylindrical shell are written as follows:

$$\begin{bmatrix}N_x\\ N_y\\ N_{xy}\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12} & A_{16}\\ A_{12} & A_{22} & A_{26}\\ A_{16} & A_{26} & A_{66}\end{bmatrix}\begin{bmatrix}\partial u/\partial x\\ \partial v/\partial y + w/R\\ \partial v/\partial x + \partial u/\partial y\end{bmatrix} \qquad (5)$$

$$\begin{bmatrix}M_x\\ M_y\\ M_{xy}\end{bmatrix} = \begin{bmatrix}D_{11} & D_{12} & D_{16}\\ D_{12} & A_{22} & D_{26}\\ D_{16} & D_{26} & D_{66}\end{bmatrix}\begin{bmatrix}-\partial^2 w/\partial x^2\\ -\partial^2 w/\partial y^2\\ -2\partial^2 w/\partial x\partial y\end{bmatrix} \qquad (6)$$

where $A_{ij}$ and $D_{ij}$ are defined as follows:

$$(A_{ij}, D_{ij}) = \int_{-h/2}^{h/2} b_{ij}(1, z^2)dz \qquad (7)$$

Upon the substitution of Equations (1), (2), (4), and (5~7) into (3), the following equations of motion are found.

$$b_{11}\frac{\partial^2 u}{\partial x^2} + 2b_{16}\frac{\partial^2 u}{\partial x\partial y} + \left(b_{66} + \frac{qR}{h}\right)\frac{\partial^2 v}{\partial x\partial y} + b_{16}\frac{\partial^2 v}{\partial x^2} + b_{66}\frac{\partial^2 u}{\partial y^2} + b_{12}\frac{\partial^2 v}{\partial x\partial y} + b_{26}\frac{\partial^2 v}{\partial y^2} + \frac{1}{R}\left(b_{12} + \frac{qR}{h}\right)\frac{\partial w}{\partial x} + b_{26}\frac{\partial w}{R\partial y} - \rho\frac{\partial^2 v}{\partial t^2} = 0 \qquad (8)$$

$$b_{22}\frac{\partial^2 v}{\partial y^2} - \frac{qR}{h}\frac{\partial^2 u}{\partial x\partial y} + 2b_{26}\frac{\partial^2 v}{\partial x\partial y} + b_{66}\frac{\partial^2 v}{\partial x^2} + b_{26}\frac{\partial^2 u}{\partial y^2} + (b_{66} + b_{12})\frac{\partial^2 u}{\partial x\partial y} + b_{16}\frac{\partial^2 u}{\partial x^2} + \frac{b_{22}}{R}\frac{\partial w}{\partial y} + \frac{b_{26}}{R}\frac{\partial w}{\partial x} - \rho\frac{\partial^2 v}{\partial t^2} = 0 \qquad (9)$$

$$\frac{1}{h}\nabla_2^4 w - \frac{qR}{h}\left(\frac{\partial^2 w}{\partial y^2} + \frac{w}{R^2}\right) + \frac{1}{R}\left(b_{12}\frac{\partial u}{\partial x} + b_{22}\frac{\partial v}{\partial y} + b_{22}\frac{w}{R} + b_{26}\frac{\partial u}{\partial y} + b_{26}\frac{\partial v}{\partial x}\right) + \rho\frac{\partial^2 w}{\partial t^2} = 0 \qquad (10)$$

where

$$\nabla_2^4 = \frac{h^3}{12}\Big[ b_{11}\frac{\partial^4}{\partial x^4} + 4b_{16}\frac{\partial^4}{\partial x^3 \partial y} + 2(b_{12}+2b_{66})\frac{\partial^4}{\partial x^2\partial y^2} + 4b_{26}\frac{\partial^4}{\partial x \partial y^3} + b_{22}\frac{\partial^4}{\partial y^4}\Big] \quad (11)$$

## 3. Method of Solution

In the present analysis, the end conditions are disregarded. The expressions for the displacements are assumed

$$u = U(t)\cos\,(mx/L - ny/R) \quad (12)$$

$$v = V(t)\cos(mx/L - ny/R) \quad (13)$$

$$w = W(t)\sin(mx/L - ny/R) \quad (14)$$

where $m/2\pi$ and $n$ are the number of waves in the axial direction and the circumferential direction, respectively. Because of the anisotropy. the displacement functions used here are different from those used for isotropic cylindrcal shells.

Substituting Equations (12), (13), and (14) into Eqs. (8~10) and denoting $\alpha = L/R$, the following three relations are established

$$\rho L^2\frac{d^2U}{dt^2} + (m^2b_{11} - 2mn\alpha b_{16} + n^2\alpha^2 b_{66})U + [(m^2b_{16} - mn\alpha(b_{66}+b_{12}) + n^2\alpha^2 b_{26})]V + (-m\alpha b_{12} + n\alpha^2 b_{26})W - (q\alpha R/h)(mnV + mW) = 0 \quad (15)$$

$$\rho L^2\frac{d^2V}{dt^2} + [n^2\alpha^2 b_{26} - mn\alpha(b_{66}+b_{12}) + m^2 b_{16}]U + (n^2\alpha^2 b_{22} - 2mn\alpha b_{26} + m^2 b_{66})V + (n\alpha^2 b_{22} - m\alpha b_{26})W + \left(\frac{qR}{h}\right)(mn\alpha)U = 0 \quad (16)$$

$$\rho L^2\frac{d^2w}{dt^2} + (-m\alpha b_{12} + n\alpha^2 b_{26})U + (n\alpha^2 b_{22} - m\alpha b_{26})V + \Big[\alpha^2 b_{22} + \frac{m^4 b_{11}}{12\alpha^2}\,\frac{h^2}{R^2} - \frac{m^3 n b_{16}}{3\alpha}\,\frac{h^2}{R^2} + \frac{1}{6}m^2n^2(b_{12}+2b_{66})\left(\frac{h^2}{R^2}\right) - \frac{1}{3}(mn^3\alpha b_{26})\left(\frac{h^2}{R^2}\right) + \frac{n^4\alpha^2 b_{22}}{12}\left(\frac{h^2}{R^2}\right)\Big]W + \left(\frac{qR}{h}\right)(n^2-1)\alpha^2 W = 0 \quad (17)$$

The above set of second-order differential equations can be written the matrix form

$$\overline{C}\frac{d^2\overline{f}}{dt^2} + \left(\overline{A} + \frac{qR}{h}\overline{B}\right)\overline{f} = 0 \quad (18)$$

where

$$\overline{f} = \begin{pmatrix} U \\ V \\ W \end{pmatrix}$$

$$\overline{C} = \rho L^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \rho L^2(\overline{I})$$

$$\overline{A} = \begin{pmatrix} b_{11}' & b_{12}' & b_{13}' \\ b_{21}' & b_{22}' & b_{23}' \\ b_{31}' & b_{32}' & b_{33}' \end{pmatrix}$$

$$b_{11}' = m^2b_{11} - 2mn\alpha b_{16} + n^2\alpha^2 b_{66}$$

$$b_{12}' = m^2b_{16} - mn\alpha(b_{66}+b_{12}) + n^2\alpha^2 b_{26}$$

$$b_{13}' = -m\alpha b_{12} + n\alpha^2 b_{26}$$

$$b_{21}' = n^2\alpha^2 b_{26} - mn\alpha(b_{66}+b_{12}) + m^2 b_{16}$$

$$b_{22}' = n^2\alpha^2 b_{22} - 2mn\alpha b_{26} + m^2 b_{66}$$

$$b_{23}' = n\alpha^2 b_{22} - m\alpha b_{26}$$

$$b_{31}' = -m\alpha b_{12} + n\alpha^2 b_{26}$$

$$b_{32}' = n\alpha^2 b_{22} - m\alpha b_{26}$$

$$b_{33}' = \alpha^2 b_{22} + \frac{m^4}{12}\,\frac{h^2 b_{11}}{R^2\alpha^2} - \frac{1}{3}m^3 n\frac{h^2}{R^2}\alpha b_{16} + \frac{1}{6}m^2n^2\frac{h^2}{R^2}(b_{12}+2b_{66}) - \frac{1}{3}mn^3\frac{h^2\alpha}{R^2}b_{26} + \frac{n^4}{12}\,\frac{\alpha^2 h^2}{R^2}b_{22} \quad (19)$$

$$\overline{B} = \begin{pmatrix} 0 & -mn\alpha & -m\alpha \\ mn\alpha & 0 & 0 \\ 0 & 0 & (n^2-1)\alpha^2 \end{pmatrix}$$

The natural frequency of the unloaded cylindrical shell is $\omega$ and can be found from the following characteristic determinant[13]

$$\overline{A} - \omega^2\overline{C} = 0 \quad (20)$$

The load is assumed

$$q(t) = q_0 + q'\cos\Omega t \quad (21)$$

where $q_0$ and $q'$ are constant, and $\Omega$ is the loading frequency. The first approximation to the dynamic stability solution is[12]

$$\left| \bar{A} + (q_0 \pm q'/2)\frac{R}{h}\bar{B} - \frac{1}{4}\Omega^2\bar{C} \right| = 0 \qquad (22)$$

The static critical load can be found from

$$\left| \bar{A} + (q_0 + q')\frac{R}{h}\bar{B} \right| = 0 \qquad (23)$$

### 4. Numerical Example and Discussion

To illustrate the application of Eq. (23), numerical evaluations are made for an isotropic and two orthotropic cylindrical shells under external pressure. The matrial properties used are shown in Table 1.

**Table 1** Material properties of the cylindrical shell.

| | Orthotropic shell | | Isotropic shell |
|---|---|---|---|
| | case I | case II | |
| $E_x$ | $4\times10^6$(psi) | $40\times10^6$(psi) | $30\times10^6$(psi) |
| $E_y$ | $40\times10^6$(psi) | $4\times10^6$(psi) | $30\times10^6$(psi) |
| $G_{xy}$ | $1.5\times10^6$(psi) | $1.5\times10^6$(psi) | $11.5384\times10^6$(psi) |
| $\nu_{xy}$ | 0.025 | 0.25 | 0.3 |
| $\nu_{yx}$ | 0.25 | 0.025 | 0.3 |

Nnmerical results are presented for the cylindrical shell with $R/h=100$ and $\alpha=2$.

The characteristic determinant, Eq. (23), yields a cubic equation of q. The cubic equation has been solved by Graeffe's Root-squaring method and the least root is selected as the critical pressure. The numerical results are given in Table 2.

**Table 2** Critical pressure load (psi).

| Orthotropic shell | | Isotropic shell |
|---|---|---|
| case I | case II | |
| 124.3992<br>(126.8827)[14] | 13.0593<br>(32.6662)[14] | 103.5307<br>(147.2081) |

The values in the parentheses are the classical pressure loads which are calculated by Donnell-type theory.[14] The difference of critical values between present study and the classical theory is based on the additional load components in tangential directions in Eq. (4). For the orthotropic cylindrical shell of case 2, the value is only about 40 per cent of the classical one. The reason is the very weak bending stiffness in the circumferential direction.

For the isotropic and orthotropic cylindrical shells under pressure loading, the additional load components are not acting and can be neglected. For the anisotropic cylindrical shell, however, the additional load components should be included because of the directional property of the anisotropic material. The present theory can be applied for the analysis of dynamic stability of anisotropic cylindrical shells under various pressure laodings.

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