Journal of the
Military Operations Resarch
Society of Korea, Vol. 7, No. 2
December, 1981

Replacement Policy for Equipments that Cumulatively Deteriorate

Deok K. Yun*
Kyung S. Park*

ABSTRACT

A replacement policy for a finite time span is proposed for the cumulative process where an item can fail only when the total amount of deterioration exceeds a prespecified failure level. The optimal deterioration limit level is determined to minimize the total cost expected per unit for a given time span. An illustrative example in case of periodically inspected reformer tubes in ammonia plant is also presented.

1. INTRODUCTION

A commonly considered replacement policy is the age replacement policy in which equipments are replaced a specified time after their installation or at failure, whichever occurs first, but the strategy of replacing equipments only when they reach a certain age has shortings to regard all same aged items as being identical, irrespectively of their current conditions on which their failure rate is dependent.

A wear dependent replacement rule has been proposed by Mercer (1961) to overcome such shortcomings of age replacement policy but

* Korea Advanced Institute of Science and Technology

this policy also has the other shortcomings to require wear monitoring system to detect continuously deterioration level and preparing a warm stand-by system; otherwise the whole system should be shutdown only to prevently replace the item whenever its deterioration level reaches a specified value.

As a mixed strategy to overcome shortcomings of age replacement and continuous inspection and wear dependent replacement policies, a periodical inspection and wear dependent replacement model is proposed for the cumulative process where an item can fail only when the total amount of deterioration exceeds a prespecified level. This model was used by Nakagawa(1976) and Derman(1960). This type of replacement policy should determine optimal inspection period and corresponding deterioration limit in consideration of economic factors. However, we confine our attention to the following chemical plant problems of determining only optimal deterioration limit for the sequential, preventive replacement rule in a finite time span, since inspection can be performed only during annual overhaul period predetermined for whole plant preventive maintenance.

In this scheme we investigate the total amount of deterioration of the nonfailed items immediately after annual operation. If the deterioration level exceeds a prespecified limit w^*, we shall replace the items before they have failed, otherwise, we shall leave it alone. During operation failures are instantly detected and temporary emergency measures are achieved at a heavy cost. Since the replacement is a time consuming process, failed items as well as nonfailed items can be newly replaced with identical ones only during annual shutdown periods.

Letting $N_1(w)_T$ denote the number of failures and $N_2(w)_T$ denote the number of replacement for a specified deterioration limit w in a given time span T, we may express the expected cost during (0,T) for a specified deterioration limit w

$$C(w)_T = A * EN_1(w)_T + a * EN_2(w)_T \tag{1}$$

where we define

A = total amount of costs resulting from equipment failure including repair cost, production loss, and raw material loss costs, but exclusive of newly replacement cost.

a = cost for newly replacing the equipment with identical one.

We shall seek the optimal deterioration limit minimizing $C(w)_T$ for a finite time span.

Let us clearly define our cumulative process of deterioration. The deterioration can be wear, fatigue, corrosion, erosion, and/or physical and chemical degradation. Assume that the amount of deterioration X_t occurring during the interval (t-1, t) is observable at discrete time points t and that the sequence X_t for t = 1,2,..., T is a sequence of independent, nonnegative, random variables with known pdfs f_1, f_2, ..., f_T. The equipment is known to fail if $\Sigma_{t=1}^{n} X_t > w_f$ (a given constant).

2. THE MATHEMATICAL MODEL

It is convenient to describe this sequential replacement model for a finite time span as a Markovian approach. Fig. (1) shows how the state space and transition diagram for this system might be constructed. States are defined simply by the equipment ages and labeled by the nonnegative integer numbers i = 0,1,2,..., T and subscript s means the states prior to inspection and preventive replacement.

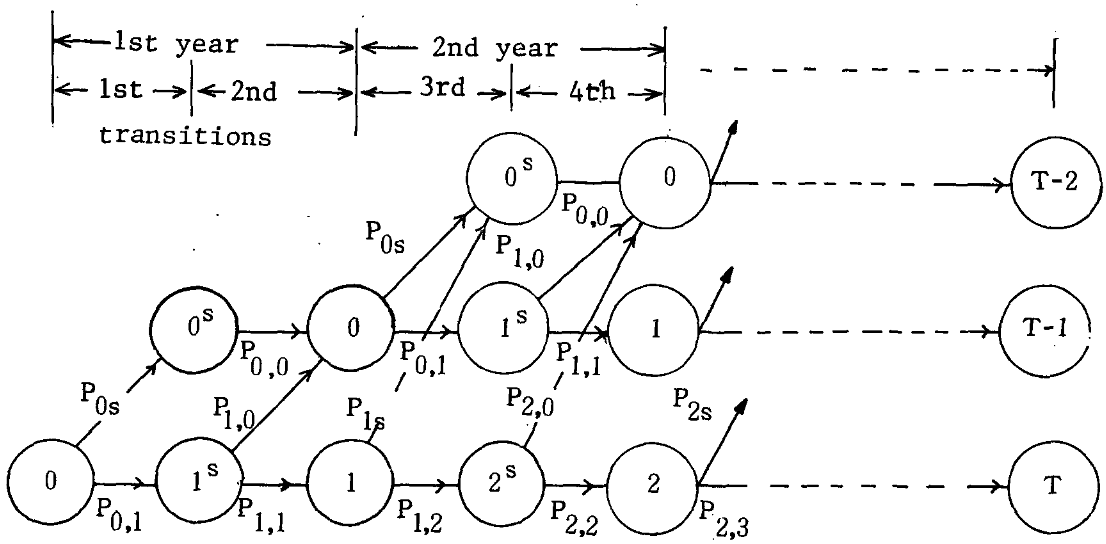

Fig. (1) Network flow model of equipment ages

During each annual operation equipments either grow a year older (transition state i to $i+1^s$) or fall into failed states (transition state i to 0^s). After each inspection and replacement all nonfailed equipments are classified into either state to leave alone (transition state i^s to i) or state to be prevently replaced (transition state i^s

to 0). We now identify state 0^s with a failed item and state 0 with new item. We can easily see that this process can be modified to a Markov chains with a finite time space $E = (0^s, 0, 1^s, 1, \ldots\ldots, T^s)$. Two times of transitions are occurred per a year. Let I_{2t-1} denote the state of the items prior to inspection and I_{2t} denote the state after t th year replacement.

In this case we can think of **P** as the transition matrix having 2T+2 rows and 2T+2 columns given by

$$\mathbf{P} = \begin{array}{c} \\ 0^s \\ 0 \\ 1^s \\ \\ T^s \end{array} \begin{pmatrix} 0^s & 0 & 1^s & & T^s \\ p(0^s,0^s) & p(0^s,0) & p(0^s,1^s) & \cdots\cdots & p(0^s,T)^s \\ p(0,0^s) & p(0,0) & p(0,1^s) & \cdots\cdots & p(0,T^s) \\ p(1^s,0^s) & p(1^s,0) & p(1^s,1^s) & \cdots\cdots & p(1^s,T^s) \\ & & & & \\ p(T^s,0^s) & p(T^s,0) & \cdots & \cdots & p(T^s,T^s) \end{pmatrix} \qquad (2)$$

Consequently using the network flow probability notations,

$$\mathbf{P} = \begin{array}{c} \\ 0^s \\ 0 \\ 1^s \\ \cdot \\ \cdot \\ T-1 \\ T^s \end{array} \begin{pmatrix} 0^s & 0 & 1^s & & & T^s \\ 0 & 1 & 0 & & & 0 \\ p_{0s} & 0 & p_{0,1} & & & 0 \\ 0 & p_{1,0} & 0 & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{T-1,0} & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & & 1 \end{pmatrix} \qquad (3)$$

In this case define for $i = 0,1,\ldots\ldots,T-1$,

p_{is} = probability the equipment of age i will fail in the interval $(i,i+1)$

p_{i0} = probability the equipment of age i will be prevently replaced in the interval $(i,i+1)$

$p_{i,i}$ = probability the equipment of age i will grow a year older

$p_{i,i+1}$ = probability the equipment of age i will successfully run in the interval $(i,i+1)$.

Then each row sum is unity. For all $i = 0,1,\ldots\ldots\ldots,T-1$,

$$\begin{aligned} p_{is} + p_{i,i+1} &= 1 \\ p_{i0} + p_{ii} &= 1 \end{aligned} \tag{4}$$

Actual values of the stationary transition probability $p(i,j)$ is the function of the nature of the unit and the original replacement rule w^*. The method to get the actual values will be discussed in the next section.

Suppose that we construct the unit with the new items which means that the initial states are 0. Then, the probabilities of being in states i^s and i respectively in the t th year are equal to probabilities to be in state i^s and i after 2t-1 and 2t times of transitions from initial state 0. If we define $\mathbf{P}^n$ to be n-step transition matrix which is the nth power of $\mathbf{P}$ matrix. Hence, we obtain,

$$p(I_{2t-1}=i^s)=p(I_{2t-1}=i^s|\, I_0=0)*p(I_0=0)=p^{(2t-1)}(0,i^s) \tag{5}$$

$$p(I_{2t}=i)=p(I_{2t}=i|\, I_0=0)*p(i_0=0)=p^{(2t)}(0,i)$$

Furthermore, the probability of failure occurring in the t th year is equal to that of being in state 0^s in the t th year.

$$p_{ft}=p^{(2t-1)}(0,0^s) \tag{6}$$

and similiarly the probability of replacement occurring in the t th year is equal to that of being state 0 in the t th year.

$$p_{rt}=p^{(2t)}(0,0) \tag{7}$$

Therefore, the cost incurred in the t th year is:

$$C_t=A*p^{(2t-1)}(0,0^s) + a*p^{(2t)}(0,0) \quad \text{for } t=1,2,\dots,T-1 \tag{8}$$

$$C_T=A*p^{(2t-1)}(0,0) \qquad \text{for } t=T$$

Letting r_t denote discounting factors of time $t = 1,2,\dots\dots,T$, then, discounted total cost incurred in the finite time span T is:

$$TC(w) = \Sigma_{t=1}^{T} C_t r_t = A* \Sigma_{t=1}^{T} p^{(2t-1)}(0,0^s)*r_t$$

$$+ a* \Sigma_{t=1}^{T} p^{(2t)}(0,0)*r_t \tag{9}$$

which is the function of the original replacement rule w. Optimization problem of this model is formulated as followings:

$$\text{Objective function: Minimize } TC(w) \quad 0 < w < w_f \tag{10}$$

$$\text{subject to } \quad 0 \le p^{(2t-1)}(0,0^s) \le \alpha \qquad \text{for safety}$$

$$0 \le p^{(2t)}(0,0) \le \beta \qquad \text{for repairing capacity}$$

Optimization might be carried through differentiating equation (9) with respect to w, but then the gradient is not readily available Fibonacci and Golden Section search techniques are appropriate since it is known that TC(w) is unimodal and a minimum of TC(w) is to located with w in the interval $0 \leq w \leq w_f$. The unimodality condition results from the fact that $p^{(2t-1)}(0,0^s)$ is monotone increasing function of w and $p^{(2t)}(0,0)$ is monotone decreasing function of w.

If the finite time span T is sufficiently large, then it may be difficult to compute the T th power of transition matrix P. It is easy to varify that the probability of failure occurring in the interval (t-1,t) satisfies the following renewal type of equation for t = 2,3,.....,T:

$$p^{(2t-1)}(0,0^s) = \Sigma_{i=1}^{t-1} p^{(2i)}(0,0) * P(X_1+X_2+.....+X_{t-i-1} < w, \quad X_1+X_2+.....+X_{t-i} > w_f) \tag{11}$$

$$p^{(2t)}(0,0) = \Sigma_{i=1}^{t-1} p^{(2i)}(0,0) * P(X_1+X_2+.....+X_{t-i-1} < w, \quad X_1+X_2+..........+X_{t-i} > w) + p^{(2t-1)}(0,0^s)$$

where X_t is the amount of deterioration occurring in the interval (t-1,t) and for t = 1

$$p(0,0^s) = P(x_1 > w_f) \tag{12}$$

$$p^{(2)}(0,0) = P(X_1 > w)$$

3. TRANSITION PROBABILITY AND MATRIX

Let us consider a simplified procedure for computation such a sequential convolution and truncation problem by the following lemma:

LEMMA. Let

X, Y : nonnegative independent random variables with pdf f_X, f_Y respectively;

X^c : X truncated at $c > 0^{(+)}$;

$Z_c = X^c + Y$;

$Z = X + Y$;

Z_c^d : Z_c truncated at $d > 0$;

Z^d : truncated at $d > 0$.

Then, Z_c^d and Z^d are identically distributed random variables even if $d \leq c$.

PROOF. By the definition of truncation, the probability density function (hereinafter pdf) of X^c is:

$$f_{X^c}(x) = \begin{cases} f_X(x)/F_X(c) & \text{for } 0 < x < c; \\ 0 & \text{otherwise}. \end{cases} \tag{13}$$

The pdf of $Z_c = X^c + Y$ is the convolution of f_{X^c} and f_Y, consequently

$$f_{Z_c}(z) = \begin{cases} \int_0^z f_{X^c}(x) \cdot f_Y(z-x)\,dx = \int_0^z f_y(z-x) f_X(x)/F_X(c)\,dx & \text{if } 0 \leq z \leq c \\ \int_0^c f_y(z-x) f_X(x)/F_X(c)\,dx & \text{if } z > c \end{cases} \tag{14}$$

Similiarly, the pdf of Z_c^d which is Z_c truncated at d is:

$$f_{Z_c^d}(z) = f_{Z_c}(z) / \int_0^d f_{Z_c}(z)dz = f_{Z_c}(z)/F_{Z_c}(d) \quad \text{for } 0 \leq z \leq d$$

$$= \frac{\int_0^z f_X(x) \cdot f_Y(z-x)dx / F_X(c)}{\int_0^d \int_0^z f_X(x) \cdot f_Y(z-x)/F_X(c)dx\, dz}$$

$$= \frac{\int_0^z f_X(x) \cdot f_Y(z-x)dx}{\int_0^d \int_0^z f_X(x) \cdot f_Y(z-x)\, dx\, dz} \qquad (15)$$

if d is less than or equal to c.

And if d is greater than c, then,

$$f_{Z_c^d}(z) = \frac{f_{Z_c}(z)}{\int_0^d f_{Z_c}(z)dz} \qquad (16)$$

$$= \frac{\int_0^z f_X(x)/F_X(c) \cdot f_Y(z-x)\, dx}{\int_0^c \int_0^z f_X(x)/F_X(c) \cdot f_Y(z-x)dx + \int_c^d \int_0^c f_X(x)/F_X(c) \cdot f_Y(z-x)dx}$$

$$= \frac{\int_0^z f_X(x) f_Y(z-x)\, dx}{\int_0^c \int_0^z f_X(x) f_Y(z-x)dx\, dz + \int_c^d \int_0^c f_X(x) f_Y(z-x)\, dx\, dz}$$

if $0 < z < c$ and,

$$f_{Z_c^d}(z) = \frac{\int_0^c f_X(x) f_Y(z-x)dx}{\int_0^c \int_0^z f_X(x) f_Y(z-x)dx\, dz + \int_c^d \int_0^c f_X(x) f_Y(z-x)\, dx\, dz}$$

if $c < z < d$. From the convolution of f_X and f_Y we get the pdf of Z:

$$f_Z(z) = \int_0^z f_X(x) f_Y(z-x)\, dx \tag{17}$$

From equation (17) and definition of truncation, the pdf of Z^d

$$f_{Z^d}(z) = \begin{cases} \dfrac{\int_0^z f_X(x) f_Y(z-x)dx}{\int_0^d \int_0^z f_X(x) f_Y(z-x)dx\, dz} & \text{if } 0 \leq z \leq d, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Comparing equation (15),(16), and (18), we arrive at the conclusion that Z_c^d and Z^d are identically distributed even if $d \leq c$. Q.E.D.

If preventive replacements are performed with the replacement rule w, then the deterioration level of the equipment after t th annual operation Y_t will be a sequence of random variables sequencially truncated at w in each interval. Hence, the following results are quite intuitive:

$$\begin{aligned} Y_1 &= X_1, && \text{for } t = 1 \\ Y_2 &= Y_1^w + X_2 && \text{for } t = 2 \\ Y_t &= Y_{t-1}^w + X_t && \text{for } t = 2,3,\ldots,T. \end{aligned} \tag{19}$$

Let $Z_t = \sum_{i=1}^t X_i$ and Z_t^w denote Z_t truncated at w, from the lemma Z_t^w and Y_t^w will be identically distributed. Therfore, the pdf of Y_t can be represented by the formula

$$f_{Y_t}(y) = f_{Z_t}(y) = \begin{cases} \int_0^y f_{Z_{t-1}^w}(x) f_{X_t}(y-x)dx, & \text{for } 0 \leq y \leq c, \\ \int_0^w f_{Z_{t-1}^w}(x) f_{X_t}(y-x)dx, & \text{for } c < y. \end{cases} \tag{20}$$

where $f_{Z^w_{t-1}}(z)$ are pdf of Z^w_{t-1}. Clearly, the transition probability p(i,j) can be determined recursively using equation (20) for i,j ϵ E. Using the notations of equation (3), entries of the transition matrix **P** will be obtained as followings:

$$p_{i-1,i} = \int_0^{w_f} \int_0^{y} f_{Z_{i-1}}(x)\ f_{X_i}(y-x)\ dx\ dy / \int_0^{w} f_{Z_{i-1}}(x)\ dx \qquad (21)$$

$$p_{i-1,s} = 1-p_{i-1,i}$$

$$p_{i,i} = \int_0^{w} \int_0^{y} f_{Z_{i-1}}(x)\ f_{X_i}(y-x)\ dx\ dy/pi-1,i$$

$$p_{i,0} = 1 - p_{i,i}$$

4 . EXAMPLE

Consider the replacement of the reformer tubes in ammonia plant of which life is limited by the technical obsolescent. The most important cause of the failure is considered as creep rupture associated with overheating. It is known that if the enlargement rate of the outside diameter exceeds w = 2.5%, the tube is regarded as creep ruptured and emergency measure to nip the pigtail of the ruptured tube is accompanied by naptha and ammonia production loss to the amount of A = ₩ 70 MM(about $100 M). Then, the usual replacement cost of the tube is a = ₩3.5 MM (about $5,000). The enlargement rate is observable every annual shutdown period and is assumed to obey the gamma distribution with scale parameter 1 and shape parameter α_t respectively for the tube age t = 1,2,...,20. The experience shows that $\alpha_1=.4$, $\alpha_2=.3$, $\alpha_3=.2$, $\alpha_4=\alpha_5=\ldots\ldots=\alpha_{17}=.1$, $\alpha_{18}=.2$, $\alpha_{19}=.3$, $\alpha_{20}=.4$. Thus we replace the tube when the anlargement rate of the tube exceeds w*, supposing that the plant useful life is 20 years. Using Fibonacci search techniques, we get the optimum enlargement rate limit w* = 1.7%.

5. CONCLUDING REMARKS

We have given a procedure for obtaining the optimum replacement policy for a finite time span. However, the usage period for some items may span an infinite interval. In this case, the optimum deterioration limit can be obtained in a modified form of this procedure. Transition matrix will be no more of use, but using Lemma, the probability of failure occurring, p_{ft} and the probability of replacement occurring can be simply obtained. As is generally true, the optimum replacement policy is more readily obtained for an infinite time sapn than for a finite time span.

REFERENCES

1. Ciuffreda, A.R. & Hopkinson, B.E. (1968), "Survey of Corrosion Problems in Reformer Hydrogen Plants," API preprint No. 13-68, 48, 33rd midyear meeting.

2. Derman, C. & Sacks, J. (1969), "Replacement of Periodically Inspected Equipment," Nav. Res. Log. Q., 597-607.

3. Mercer, A. (1961), "Some Simple Wear Dependent Renewal Process," J.R. Statist. Soc., B. 23, 368-376.

4. Morey, R.C. (1966), "Some Stochastic Properties of a Compound Renewal Damage Model," Ops Res., 14, 902-908.

5. Nakagawa, Toshio (1976), "On a Replacement Problem of a Cumulative Damage Model," Opl Res. Q., 27, 895-900.

6. Roll, Y. & Naor, P. (1968), "Preventive Maintenance of Equipment Subject to Continuous Deterioration and Stochastic Failure," Opl Res. Q., 19, 61-71.

7. Silvazlian, B.D. & Mahoney, J.F. (1978), "Group Replacement of a Multi-Component System which is Subject to Deterioration only," Adv. Appl. Prob., 10, 867-885.

8. Woodman, R.C. (1967), "Replacement Policies for Components that Deteriorate," Opl Res. Q., 18, 267-280.

9. Zeis, L.A. & Heinz, E. (1970), "Catalyst Tubes in Primary Reformer Furnaces," Chem. Eng. Prog., 60, 68-73.