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A NOTE ON CLOSENESS SPACES

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ABSTRACT

Kasahara는 任意의 集合에 Closeness Structure를 導入하여 Convergence Structure와의 關係를 밝혔는데 本 論文에서는 Closeness 空間 (X,Γ)의 部分集合 Y가 X上의 Closeness Structure Γ에 對한 相對 Closeness Structure를 갖기 爲한 條件 및 Closeness 部分空間과 Convergence 部分空間과의 關係를 考察하였다.

The purpose of this note is to introduce the closeness structure and to consider the hereditary properties of a closeness space.

In section 1, after stating several elementary properties of operations on the power set of a set, we shall introduce the closeness structure which was considered by Kasahara [1]. Therefore, some theorems in section 1 would be found in [1]. In section 2, we shall consider the subspace of a closeness space.

Throughout this note, the power set of a set X will be denoted by $P(X)$ and the value of a mapping $\alpha:P(X)\to P(X)$ at $A\in P(X)$ by A^α. The complement of $A\in P(X)$ in X will be written A^c. A mapping $\alpha:P(X)\to P(X)$ is called *monotone* if $A\subset B$ implies $A^\alpha\subset B^\alpha$ for every $A,B\in P(X)$.

1. Throughout this section, X denotes an arbitrary set. Let $\alpha:P(X)\to P(X)$ be a mapping. For each $x\in X$, we put

$$\Phi_\alpha(x)=\{A\in P(X)\,|\,x\notin A^{c\alpha}\}.$$

Evidently Φ_α is a mapping of X into $P(P(X)=PP(X)$ and the following two statements hold:

(1) $\Phi_\alpha(x)\neq\phi$ if and only if $x\notin\bigcap\{A^\alpha\,|\,A\in P(X)$.

(2) $\phi\notin\Phi_\alpha(x)$ if and only if $x\in X^\alpha$.

Moreover, if α is monotone then the following two statements hold:

(3) $x\in\{x\}^\alpha$ if and only if $A\subset A^\alpha$ for every $A\in P(X)$.

(4) If $A\in P(X)$ then $A^\alpha=\{x\in X\,|\,S\cap A\neq\phi$ for every $S\in\Phi_\alpha(x)\}$.

Let Ψ: $X \to PP(X)$ be a mapping. For each $A \in P(X)$, we put

$$A^{k(\Psi)} = \{x \in X \mid S \cap A \neq \phi \text{ for every } S \in \Psi(x)\}.$$

Evidently $k(\Psi)$ is a monotone mapping of $P(X)$ into itself. And as an immediate consequence of (4), we have the following statement:

(5) If $\alpha: P(X) \to P(X)$ is monotone then $\alpha = k(\Phi_\alpha)$.

For every subset T of $P(X)$, we put

$$[T = \{A \in P(X) \mid A \text{ contains at least one member of } T\}.$$

Then by (4) and (5), we have the following two statements:

(6) Let Ψ: $X \to PP(X)$ be a mapping, then

$$\Phi_{k(\Psi)}(x) = [\Psi(x)] \quad \text{for every } x \in X.$$

(7) If $\alpha: P(X) \to P(X)$ is monotone, then

$$[\Phi_\alpha(x)] = \Phi_\alpha(x) \quad \text{for every } x \in X.$$

DEFINITION. A mapping $\alpha: P(X) \to P(X)$ is called a *semiclosure* on X if it satisfies the following conditions:

(a) $\phi^\alpha = \phi$ and $X^\alpha = X$.

(b) $(A \cup B)^\alpha = A^\alpha \cup B^\alpha$ for every $A, B \in P(X)$.

THEOREM 1 If $\alpha: P(X) \to P(X)$ is a semiclosure on X, then $\Phi(x)$ is a filter on X for each $x \in X$.

PROOF. Let $x \in X$. Then $x \notin \phi^\alpha$. Hence by (1), $\Phi_\alpha(x) \neq \phi$; by (2), $\phi \notin \Phi(x)$. On the other hand, if $A \subset B$ where A, $B \in P(X)$, then $A^\alpha \cup B^\alpha = B^\alpha$. Hence α is monotone. Thus by (7), $[\Phi_\alpha(x)] = \Phi_\alpha(x)$. Now if $A, B \in \Phi_\alpha(x)$. then since $x \notin A^{c\alpha}$ and $x \notin B^{c\alpha}$, we have

$$x \notin A^{c\alpha} \cup B^{c\alpha} = (A^c \cup B^c)^\alpha = (A \cap B)^{c\alpha},$$

which shows that $A \cap B \in \Phi_\alpha(x)$. This completes the proof.

DEFINITION. Let Γ be a set of semiclosures on a set X. The ordered pair (X, Γ) is called a *closeness space*, and Γ is called a *closeness* on X if the following conditions are satisfied:

(C1) For every $x \in X$, there exists an $\alpha \in \Gamma$ such that $x \in \{x\}^\alpha$.

(C2) For every $\alpha, \beta \in \Gamma$, there exists a $\gamma \in \Gamma$ such that $A^\alpha \cup A^\beta \subset A^\gamma$ for every $A \in P(X)$.

Let Γ, Γ' be two closeness on a set X. We say that Γ' is finer than Γ (or Γ is coarser than Γ') if for every $x \in X$ and for every $\alpha \in \Gamma$, there exists a $\beta \in \Gamma'$ such that $\Phi_\beta(x) \subset \Phi_\alpha(x)$.

By THEOREM 1, we have the following theorem (theorem 3 in [1]), which shows that every convergence structure can be descrived by a closeness structure.

THEOREM 2. Let X be a set. For each closeness Γ on X, there exists a convergence structure τ on X such that, for every $x \in X$, $\mathscr{V} \in \tau(x)$ if and only if $\Phi_\alpha(x) \subset \mathscr{V}$ for some $\alpha \in \Gamma$.

2. Let Y be a subset of a set X. For any semiclosure $\alpha : P(X) \to P(X)$, define a mapping $\alpha' : P(Y) \to P(Y)$ by taking

$$A^{\alpha'} = A^\alpha \cap Y \quad \text{for every } A \in P(Y).$$

Then α' need not to be a semiclosure on Y, but the following lemma holds:

LEMMA 3. Let Y be a subset of a set X. If $\alpha : P(X) \to P(X)$ is a semiclosure on X such that $A \subset A^\alpha$ for every $A \in P(X)$. Then $\alpha' : P(Y) \to P(Y)$ is a semiclosure on Y.

PROOF. Clearly $\phi^{\alpha'} = \phi$. Since $Y \subset Y^\alpha$, $Y^{\alpha'} = Y^\alpha \cap Y = Y$. On the other hand, for every $A, B \in P(Y)$, we have

$$(A \cup B)^{\alpha'} = (A \cup B)^\alpha \cap Y = (A^\alpha \cup B^\alpha) \cap Y = (A^\alpha \cap Y) \cup (B^\alpha \cap Y) = A^{\alpha'} \cup B^{\alpha'}.$$

This completes the proof.

Let Y be a subset of a set X. For any closeness Γ on X, we put

$$\Gamma_Y = \{\alpha' : P(Y) \to P(Y) \mid \alpha \in \Gamma\}.$$

Then we have

THEOREM 4. Let Y be a subset of a set X. Let Γ be a closeness on X such that for each $\alpha \in \Gamma$ and for each $A \in \mathrm{P}(X)$, $A \subset A^\alpha$. Then Γ_Y is a closeness on Y.

PROOF. By LEEMA 3, Γ_Y is a set of semiclosures on Y. By (3), Γ_Y satisfies(C1). On the other hand, for every $\alpha', \beta' \in \Gamma_Y$, there exist $\alpha, \beta \in \Gamma$ such that $A^{\alpha'} = A^\alpha \cup Y$ and $A^{\beta'} = A^\beta \cap Y$ for every $A \in P(Y)$. Hence there exists a $\gamma \in \Gamma$ such that $A^\alpha \cup A^\beta \subset A^\gamma$ for every $A \in P(X)$. Thus there exists a $\gamma' \in \Gamma_Y$ such that $A^{\gamma'} = A^\gamma \cap Y$ for every $A \in P(Y)$. And we have

$$A^{\alpha'} \cup A^{\beta'} = (A^\alpha \cap Y) \cup (A^\beta \cap Y) = (A^\alpha \cup A^\beta) \cap Y \subset A^\gamma \cap Y = A^{\gamma'}.$$

This completes the proof.

This Γ_Y will be called the relative closeness on Y with respect to Γ. The closeness space (Y, Γ_Y) will be called a subspace of (X, Γ).

As an immediate consequence of THEOREM 2 and 4, we have

THEOREM 5. Let (Y, Γ_Y) be a subspace of a closeness space (X, Γ). Let τ, τ_Y be the convergence structures induced by Γ, Γ_Y, respectively. If τ'_Y is the relative convergence structure on Y with respect to τ, then $\tau_Y = \tau'_Y$.

REFERENCE

〔1〕. Shouro Kasahara, *Closeness Spaces and Convergence Spaces*, Proc. of Japan Academy, Vol. 50, No.4 (1974).

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