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ON THE INTEGRAL THEORY OVER DIFFERENTIABLE MANIFOLDS* (II)

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ABSTRACT

論文〔3〕(本 論文 第1部)에서 微分可能多樣體 M 위의 $(n-1)$次 微分型式 $\beta^{(n-1)}$이 Compact인 Carrier를 가지면 $\int d\beta^{(n-1)}=0$이며, $(p-1)$次 微分型式 $\beta^{(p-1)}$과 p次 微分可能鎖 $C^{(p)}=\sum_i k_i S_i^{(p)}$에 對하여 $\int_{C^{(p)}} d\beta^{(p-1)}=\int_{\partial C^{(p)}} \beta^{(p-1)}$이 成立 (Stokes 定理의 一般化)…等 M위의 積分에 관한 여러가지 性質들을 究明하였다. 이 性質들을 土台로 하여 本 論文에서는; 第2節에서 微分可能多樣體 M위의 Lie 導函數의 定義와 Lie微分에 關한 여러가지 性質들을 考察하고,

第3節에서 $div\ X$와 Laplace 作用素 $\triangle f$의 定義 및 實 n次元 可符號微分可能 多樣體 M 위에서의 $divX$와 $\triangle f$의 積分에 關한 性質, 即

$V=\sqrt{|g|}dx^1\Lambda\cdots\Lambda dx^n \in A^n(M)$에 對하여

$$\int_M divXV=\int_M \triangle f v=0$$

인 關係가 成立함을 究明한다. (定理 3.3)

1. Introduction

Throughout this paper, by M we mean a real n-dimensional differentiable manifold, which is paracompact. Furthermore, we put

(i) $T(M)x$ =the tangent space on $x\in M$.

(ii) $T(M)$ =the total tangent space of M.

(iii) $T^*(M)x$=the dual space of $T(M)x$

(iv) $T^*(M)$ =the dual space of $T(M)$

(v) $\mathfrak{X}(M)$ =the set of all vector fields over M.

(vi) $A^p(M)$ $=\Gamma\{A^pT^*(M)\}$, where for a vector bundle

$\xi=(E,P,X)$, $\Gamma(E)$ is the set of all cross sections of ξ.

In 〔3〕, we have already proved some properties with respect to the integral over differentiable manifolds. The purpose of this paper is to introduce the Lie derivatives

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on the differentiable manifolds and for a vector field X on a differentiable manifold to prove that the integral of $divX$, which is the divergence of X with respect to the given Riemannian matrix of a differentiable manifold, is zero.

It will be illustrated in the second section of this paper the definition of Lie derivatives and proved some properties of Lie differential. Finally, we shall prove in the third section some properties of $div\ X$(Theorem 3.2) and cur main theorem

$$\int_M divX \cdot V = \int_M \triangle fv = 0$$

for $V = \sqrt{|g|}\ dx' \Lambda \cdots\cdots \Lambda dx^n \in A^n(M)$ (Theorem 3.3).

2. Lie Derivatives

For a manifold M we put

$$T(a,b) = \underbrace{T(M)\otimes\cdots\cdots\otimes T(M)}_{\text{a-times}} \otimes \underbrace{T^*(M)\otimes\cdots\cdots\otimes T^*(M)}_{\text{b-times}}.$$

Then $T(a,b)$ is said to be *a vector bundle of (a,b)-type*, and each cross section of $T(a,b)$ is called *a tensor field of (a,b)-type.*

Let $\{U, (x^1 \cdots\cdots x^n)\}$ be a locally coordinate neighborhood of M. Then $T(M)\mid U$ has the locally basis $\left\{\frac{\partial}{\partial x^1}\cdots\cdots, \frac{\partial}{\partial x^n}\right\}$ and $T^*(M)\mid U$ has the locally basis $\{dx^1\cdots\cdots dx^n\}$ which is the dual basis of $\left\{\frac{\partial}{\partial x^1}\cdots\cdots, \frac{\partial}{\partial x^n}\right\}$. In this case, for each $\xi \in \Gamma\{T(a,b)\}$ we can put such that

$$\xi = \Sigma\ \xi^{\mu_1 \ldots \mu_a}\ V_1 \cdots V_b \frac{\partial}{\partial x^{\mu_1}}, \otimes\cdots\otimes \frac{\partial}{\partial x^{\mu_a}} \otimes dx^{V_1} \otimes\cdots\otimes dx^{V_b}$$

〔DEFINITION 2.1〕 Let ξ be a tensor field of (a,b)-type. For any $W_1, \cdots W_a \in A^1(M)$ and $X_1, \cdots\cdots, X_b \in \mathfrak{X}(M)$, and for each $x \in M$ we define

$$\xi\ (w_1 \cdots\cdots w_a,\ X_1 \cdots X_b)(x) = <\xi(x),\ w_1(x)\otimes\cdots\otimes w_a(x)\ \otimes X_1.(x)\ \otimes\cdots\otimes X_b(x)>,$$

where for $\xi,\ \eta \in T(M)$, $<\xi,\eta>$ is the inner product of ξ and η.

(Note)

$$<\frac{\partial}{\partial x^{\mu_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{\mu_a}}\otimes dx^{v_1}\otimes\cdots\otimes dx^{v_b},\ dx^{m_1}\otimes\cdots\otimes dx^{m_a}\otimes\frac{\partial}{\partial x^{\mu_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{\mu_b}}>$$

$$= <\frac{\partial}{\partial x^{\mu_1}} dx^{m_1}> \cdots <\frac{\partial}{\partial x^{\mu_a}} dx^{m_a}> <\frac{\partial}{\partial x^{\mu_1}},\ dx^{v_1}> \cdots <\frac{\partial}{\partial x^{\mu_b}},\ dx^{v_b}>$$

$$= \delta_{\mu_1 m_1}\ \delta_{\mu_2 m_2} \cdots\cdots \delta_{\mu_a m_a}\ \delta_{n_1 v_1}\ \delta_{n_2 v_2} \cdots\cdots \delta_{n_6 v_6}.$$

We can easily prove from the definition 2.1 that ξ is a multiplier map as a $C(M)$-module and the converse is true, where $C(M)$ is the set of all C^∞-functions from M to R. For each $X\in\mathfrak{X}(M)$ a linear map

$$L_X:\ \Gamma\{T(a,b)\}\longrightarrow\Gamma\{T(a,b)\}$$

is defined as follows:

(i) The case: $a=b=0$.

Since $\Gamma\{T(a,b)\}=C(M)$, for $f\in C(M)$ we define $L_Xf=X(f)$, where

$$X(f)(x)=X(x)f=\sum_{\mu=1}^{n}X^{\mu}(x)\frac{\partial f}{\partial x^{\mu}}\ \text{ for }\ X(x)=\sum_{\mu=1}^{n}X^{\mu}(x)\frac{\partial}{\partial x^{\mu}}\ .$$

(ii) The case: $a=1$ and $b=0$.

Since $\Gamma\{T(a,b)\}=\mathfrak{X}\{M\}$, for $Y\in\mathfrak{X}(M)$ we define $L_XY=[X,Y]$.

(iii) The case: $a=0$ and $b=1$.

Sine $\Gamma\{T(a,b)\}=A^1(M)$, for $\xi\in A^1(M)$ we define

$(L_X\xi)(Y)=X\{\xi(Y)\}-\xi\{[X,Y]\}$ for $Y\in\mathfrak{X}$.

In the above cases it is clear that L_X is a linear map from $\Gamma\{T(a,b)\}$ to itself. That is, the following is obvious:

(a) $L_X(f,g)=fL_Xg+L_Xf\cdot g$ $\quad(f,g\in C(M))$

(b) $L_X(f,Y)=fL_XY+(L_Xf)\cdot Y$ $\quad(f\in C(M).\ Y\in\mathfrak{X}(M)(2.1)$

(c) $L_X(f,\xi)=fL_X\xi+(L_Xf)\cdot\xi$ $\quad(f\in C(M),\ \xi\in A^1(M))$

[PROPOSITION 2.2] $L_{[X,Y]}=L_X\cdot L_Y-L_Y\cdot L_X$.

PROOF. It suffices to prove the case (iii).

For $\xi\in A^1(M)$ and $Z\in\mathfrak{X}(M)$ we have

$$\begin{aligned}\{L_{[X,Y]}\xi\}(Z)&=[X,Y]\xi(Z)-\xi\{[X,Y],Z\}\\&=X\{Y\xi(Z)\}-Y\{X\xi(Z)\}-\xi\{[X,[Y,Z]]\}+\xi\{[Y,[X,Z]]\}\\&=(L_X\cdot L_Y\xi)(Z)-(L_Y\cdot L_X\xi)(Z).\end{aligned}$$

[DEFINITION 2.3] Given a vector field $X \in \mathfrak{X}(M)$ we define that for a tensor field $\xi \in \Gamma\{T(a,b)\}$

$$(L_X\xi)(w_1\cdots\cdots w_a,\ X_1\cdots\cdots X_b)$$

$$=L_X\{\xi(w_1\cdots\cdots w_a,\ X_1\cdots\cdots X_b\}-\sum_{1=s}^{a}\xi(w_1\cdots\cdots L_Xw_s\cdots\cdots w_a,\ X_1\cdots\cdots X_b)$$

$$-\sum_{t=1}^{b}\xi(w_1\cdots\cdots w_a,\ X_1\cdots\cdots L_X\ X_t\cdots\cdots X_b),$$

where $w_1,\cdots\cdots w_a \in A^1(M)$ and $X_1,\cdots\cdots,X_b \in \mathfrak{X}(M)$.
In this case, $L_X\xi$ is said to be *the Lie derivative of ξ with respect to X.*

[PROPOSITION 2.4]

(i) $L_X:\Gamma\{T(a,\mathrm{b})\}\longrightarrow\Gamma\{T(a,b)\}$ is a linear map.

(ii) For $\xi\in\Gamma\{T(a,b)\}$ and $\eta\in\Gamma\{T(a,b)\}$ $L_X(\xi\otimes\eta)=L_X\xi\otimes\eta+\xi\otimes L_X\eta$.

(iii) $L_{[X,Y]}=L_X\cdot L_Y-L_Y\cdot L_X$.

PROOF. (i) For $w_1\otimes\cdots\otimes w_a\otimes X_1\otimes\cdots\otimes X_b\in \underbrace{A^1(M)\otimes\cdots\otimes A_1(M)}_{\text{a-times}}\otimes\underbrace{\mathfrak{X}(M)\otimes\cdots\otimes\mathfrak{X}(M)}_{\text{b-times}}$

and $f\in C(M)$, we easily see the following by the expression (2-1).

$$(L_X\xi)(w_1\cdots\cdots fw_s,\cdots\cdots w_a,\ X_1\cdots\cdots X_b)$$
$$=(L_X\xi)(w_1\cdots\cdots w_a,X_1\cdots\cdots fX_t\cdots\cdots X_b)$$
$$=f\cdot(L_X\xi)(w_1\cdots\cdots w_a,\ X_1\cdots\cdots X_b).$$

(ii) Sine $(\xi\otimes\eta)(w_1\cdots w_a,\ w'_1,\cdots w'_{a'},\ X_1\cdots X_b,\ X'_1\cdots X'_{b'})$

$$=\xi(w_1\cdots w_a,\ X_1\cdots X_b)\cdot\eta(w'_1\cdots w'_a,\ X'_1\cdots X'_{b'})$$

for $\xi\in\Gamma\{T(a,b)\}$ and $\eta\in\Gamma\{T(a,b]\}$ we have

$$L_X(\xi\otimes\eta)\ (w_1\cdots w_a,\ w'_1\cdots w'_{a'}\ X_1\cdots X_b,\ X'_1\cdots X'_{b'})$$
$$=L_X(\xi(w_1\cdots w_a,\ X_1\cdots X_b))\cdot\eta(w'_1\cdots w'_{a'},\ X'_1\cdots X'_{b'})$$
$$-\sum_{s=1}^{a}\xi(w_1\cdots L_X\ w_s,\cdots w_a,\ X_1\cdots X_b)\cdot\eta(w'_1\cdots w'_{a'},\ X'_1\cdots X'_{b'})$$
$$-\sum_{t=1}^{b}\xi(w_1\cdots w_a,\ X_1\cdots L_X\ X_t\cdots X_b)\cdot\eta(w'_1\cdots w'_a,\ X'_1\cdots X'_{b'})$$
$$+\xi(w_1\cdots w_a,\ X_1\cdots X_b)\ L_X(\eta(w'_1\cdots w'_{a'},\ X'_1\cdots X'_{b'}))$$
$$-\xi(w_1\cdots w_a,\ X_1\cdots X_b)\ \sum_{s'=1}^{a'}\eta(w'_1\cdots L_Xw'_{s'}\cdots w'_{a'},\ X'_1\cdots X'_{b'})$$
$$-\xi(w_1\cdots w_a,\ X_1\cdots X_b)\ \sum_{t'=1}^{b'}\eta(w'_1\cdots w'_{a'},\ X'_1\cdots L_X\ X'_{t'}\cdots X'_{b'})$$

$$=(L_X\xi\otimes\eta)(w_1\cdots w_a,\ w'_1\cdots w'_{b'},\ X_1\cdots X_b,\ X'_1\cdots X'_{b'})$$
$$+(\xi\otimes L_X\eta)(w_1\cdots w_a,\ w'_1\cdots w'_{a'},\ X_1\cdots X_b,\ X'_1\cdots X'_{b'})$$
$$=(L_X\xi\otimes\eta+\xi\otimes L_X\eta)(w_1\cdots w_a,\ w'_1\cdots w'_{a'},\ X_1\cdots X_b,\ X'_1\cdots X'_{b'}.)$$

(iii) For $\xi\in\Gamma\{T(a,b)\}$ we may assume using a partition of unity that the carrier of ξ is in a coordinate neighborhood U as a compact set.

Then ξ is a linear combination of forms

$$X_1\otimes\cdots\cdots\otimes X_a\cdots\cdots\otimes\eta_1\otimes\cdots\cdots\otimes\eta_b,$$

where X_i $(i=1,2,\cdots,a)\in\mathfrak{X}(M)$ and $\eta_j(=1,2,\cdots,b)\in A^1(M)$ have their carriers in U.

From (ii), since $L_{(X,Y)}(X_1\otimes\cdots X_a\otimes\eta_1\otimes\cdots\otimes\eta_b)$

$$=L_{[X,Y]}X_1\otimes(X_2\otimes\cdots\otimes X_a\otimes\eta_1\otimes\cdots\otimes\eta_b)$$
$$+X_1\otimes L_{[X,Y]}(_{X,Y}X_2\otimes\cdots\otimes X_a\otimes\eta_1\otimes\cdots\otimes\eta_b)\qquad\text{(by proposition 4.2)}$$
$$=(L_X\cdot L_Y-L_Y\cdot L_X)\ (X_1\otimes X_2\otimes\cdots\otimes X_a\otimes\eta_1\otimes\cdots\otimes\eta_b)$$
$$+X_1\otimes(L_{[X-Y]}X_2)\otimes\ (X_3\otimes\cdots\otimes X_a\otimes\eta_1\otimes\cdots\otimes\eta_b)$$
$$+X_2\otimes L_{[X,Y]}(X_3\otimes\cdots\otimes X_a\otimes\eta_1\otimes\cdots\otimes\eta_b)$$
$$=\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$=(L_X\cdot L_Y-L_Y\cdot L_X)\ (X_1\otimes\cdots\otimes X_a\otimes\eta_1\otimes\cdots\otimes\eta_b),,$$

we have

$$L_{[X,Y)}=L_X\cdot L_Y-L_Y\cdot L_X.$$ Q.E.D.

For $X\in\mathfrak{X}(M)$ and $\xi\in A_1(M)$ we define

$$\{i(X)\xi\}(X_1\cdots\cdots X_{p-1})=\xi(X,X_1\cdots\cdots X_{p-1}),$$

where $X_1,\cdots X_{p-1}\in\mathfrak{X}(M)$. That is,

$$i(X):\Gamma\{T(0,p)\}\longrightarrow\Gamma\{T(0,p-1)\}$$
$$\xi\ \longmapsto i(X)\xi.$$

[PROPOSITION 2.] For $\xi\in A^p(M)$ we have

$$L_X\xi=i(X)d\xi+di(X)\xi.$$

PROOF. FOR $X_1\cdots\cdots X_p\in\mathfrak{X}(M)$

$$\{i(X)d\xi\}(X_1\cdots X_p)=d\xi(X,X_1\cdots X_p)$$
$$=X\{\xi(X_1\cdots X_p)\}+\sum_{i=1}^{p}(-1)^i X_i\{\xi(X,X_1\cdots \hat{X}_i\cdots X_p)\}$$
$$+\sum_{j=1}^{p}(-1)^{1+j+1}\xi([X,X_j],\ \hat{X},\cdots\hat{X}_j\cdots X_p)$$
$$+\sum_{i<j}(-1)^{(i+1)+(j+1)}\xi([X_i,X_j],\ X,\cdots\hat{X}_i\cdots\hat{X}_j\cdots X_p)$$
$$=X\{\xi(X_1\cdots X_p)\}+\sum_{j=1}^{p}(-1)^{1+j+1}\xi([X,X_j],\ X_1\cdots X_j\cdots X_p)$$
$$+\sum_{i=1}^{p}(-1)^i X_i\{\xi X,\hat{X}_1\cdots\hat{X}_i\cdots X_p)\}+\sum_{i<j}(-1)^{(i+1)+(i+1)}\xi([X_i,X_j],X_1,\cdots\hat{X}_i\cdots\hat{X}_j\cdots X_p)$$
$$=(L_X\xi)(X_1\cdots X_p)-\{di(X)\xi\}(X_1\cdots X_p).$$

Therefore, we have

$$L_X\xi=i(X)d\xi+di(X)\xi.$$ Q.E.D.

3. Divergences and Laplacian Operators

Since a manifold M has a Riemannian metrix g, we put

$$g^a_{\mu\nu}(X)=\left(\frac{\partial}{\partial x^\mu_a},\ \frac{\partial}{\partial x^\nu_a}\right)_x\quad(\mu,v=1,2,\cdots\cdots,n)$$

on a locally coordinate neighborhood $\{U_a,\ (x^1_a,\cdots,x^n_a)\}$ of M. Thus, $g^a_{\mu\nu}(X)$ is an inner product of $\frac{\partial}{\partial x^\mu_a}$ and $\frac{\partial}{\partial x^\nu_a}$ in $T(M)x$.

Put $det\ (g^{\mu\nu}_a)=(g^{\mu\nu}_a)=(g^a_{\mu\nu}{}^{-1})=(det\,(g^a_{\mu\nu}))^{-1}$,

then $g^{\mu\nu}_a$ is an inner product of dx^μ_a and dx^ν_a in $T^*(M)x$.

The cannonical isomorphism form $\Gamma\{T(M)\}$ to $\Gamma\{T^*(M)\}$ is define by the following maps. For each locally coordinate neighborhood $\{U,(x^1\cdots x^n)\}$ of M,

$$\Gamma\{T(M)|U\}\ni\sum_{\mu=1}^{n}\alpha^\mu(x)\frac{\partial}{\partial x^\mu}\ \longmapsto\sum_{\mu=1}^{n}\left\{\sum_{\nu=1}^{n}g_{\mu\nu}(x)\alpha^\nu(x)\right\}dx^\mu\in\Gamma\{T^*(M)|U\}.$$

and

$$\Gamma\{T^*(M)|U\}\ni\sum_{\mu=1}^{n}\beta_\mu(x)dx^\mu\ \longmapsto\sum_{\mu=1}^{n}\left\{\sum_{\nu=1}^{n}g^{\mu\nu}(x)\beta_\nu(x)\right\}\in\Gamma\{T(M)|U\}.$$

Sine $df\in\Gamma\{T^*(M)\}$ for $f\in C(M)$ we have by the above map

$$\begin{array}{ccc}\Gamma\{T^*(M)\} & \longrightarrow & \Gamma\{T(M)\}\\ \cup & & \cup\\ df & \longmapsto & \sum_{\nu=1}^{n}g^{\mu\nu}\frac{\partial f}{\partial x^\nu}\cdot\frac{\partial}{\partial x^\mu}\end{array}\qquad(3\text{-}1)$$

because of that $df=\sum_{\nu=1}^{n}\frac{\partial f}{\partial x^\nu}dx^\nu$.

[DEFINITION 3.1] For $V=[det(g^{\mu\nu})]^{\frac{1}{2}}dx^1\Lambda\cdots\Lambda dx^n \in A(M)$ we define

$$L_XV=divXV, \quad \text{where } X\in\mathfrak{X}(M)$$

Since $L_XU\in A^n(M)$ we have $divX\in C(M)$. In this case, $divX$ is called the divergence of X with respect to the given Riemannian matrix $g^{\mu\nu}$. We put

$$\sum_{\nu=1}^{n} g^{\mu\nu}\frac{\partial f}{\partial x^\nu}=grad\ f$$

which is called the gradient of f. We also put

$$\triangle f=div\ grad\ f,$$

which is called the Laplacian operator of f.

[THEOREM 3.2] If we put $det(g_{\mu\nu})=|g|$, $X=\sum_{\mu=1}^{n} X^\mu\frac{\partial}{\partial x^\mu}$,

then we have

$$divX=\sum_{\mu=1}^{n}\frac{\partial X^\mu}{\partial x^\mu}+\frac{1}{2}\sum_{\mu=1}^{n} X^\mu\frac{\partial log|g|}{\partial x^\mu},$$

$$\triangle f=\sum_{\mu,\nu=1}^{n} g^{\mu\nu}\frac{\partial^2 f}{\partial x^\mu\partial x^\nu}+\sum_{\mu,\nu=1}^{n}\left(\frac{\partial g^{\mu\nu}}{\partial x^\mu}+\frac{1}{2}g^{\mu\nu}\frac{\partial log|g|}{\partial x^\mu}\right)\frac{\partial f}{\partial x^\nu}.$$

PROOP. Since $V=\sqrt{|g|}dx^1\Lambda\cdots\Lambda dx^n$, for $X_1\cdots X_n\in\mathfrak{X}(M)$ we have

$$L_XV(X_1\cdots X_n)=L_X\{V(X_1\cdots X_n)\}-\sum_{i=1}^{n}V(X_1\cdots L_XX_j\cdots X_n)$$

$$=X\{\sqrt{|g|}det(X_i,dx^j)\}-\sum_{i=1}^{n}V(X_1\cdots[X_1X_i]\cdots X_n)$$

$$=\sum_{\mu=1}^{n}X^\mu\frac{\partial}{\partial x^\mu}\sqrt{|g|}\ det(X_i,dx^j)+\sum_{\mu=1}^{n}\sqrt{|g|}\frac{\partial X^\mu}{\partial x^\mu}det(X_i,dx^j)$$

$$=\left(\frac{1}{2}\sum_{\mu=1}^{n}X^\mu\frac{\partial log|g|}{\partial x^\mu}+\sum_{\mu=1}^{n}\frac{\partial X^\mu}{\partial X^\mu}\right)\sqrt{|g|}\ det\{X_i,dx^j\}$$

$$=divX\cdot V(X_1\cdots X_n),$$

which implies that

$$divX=\sum_{\mu=1}^{n}\frac{\partial X^{\mu}}{\partial x^{\mu}}+\frac{1}{2}\sum_{\mu=1}^{n}X^{\mu}\frac{\partial\ log|g|}{\partial x^{\mu}}.$$

Using the expression (3-1) above, we have

$$\triangle f=div\ grad\ f=div\left(\sum_{\nu=1}^{n}g^{\mu\nu}\frac{\partial f}{\partial xv}\ \frac{\partial}{\partial x^{\mu}}\right)$$

$$=\sum_{\mu=1}^{n}\frac{\partial}{\partial x^{\mu}}\left(\sum_{\nu=1}^{n}g^{\mu\nu}\frac{\partial f}{\partial x^{\nu}}\right)+\frac{1}{2}\sum_{\mu=1}^{n}\left(\sum_{\nu=1}^{n}g^{\mu\nu}\frac{\partial f}{\partial x^{\nu}}\right)\frac{\partial log|g|}{\partial x^{\mu}}$$

$$=\sum_{\mu,\nu=1}^{n}g^{\mu\nu}\frac{\partial^2 f}{\partial x^{\mu}\partial x^{\nu}}+\sum_{\mu,\nu=1}^{n}\left(\frac{\partial g^{\mu\nu}}{\partial x^{\mu}}+\frac{1}{2}g^{\mu\nu}\frac{\partial log|g|}{\partial x^{\mu}}\right)\frac{\partial f}{\partial x^{\nu}}.$$

Q.E.D.

[THEOREM 3.3] For $V=\sqrt{|g|}\ dx^1\Lambda\cdots\Lambda dx_n \in A^n(M)$,
if M is orientable then

$$\int_M divX\cdot\nu=\int_M\triangle f\cdot v.=0$$

PROOF For $\nu=\sqrt{|g|}\ dx'\Lambda\cdots\Lambda dx^n$ wc have

$$L_X\nu=divX\cdot\nu=i(X)dv+d(i(X)v). \qquad \text{(proposition 2.5)}$$

From $dv=0$ we have

$$divX\cdot\nu=L_X\nu=d(i(X)\nu).$$

Since M is orientable, by Theorem 8 in [3], we have

$$\int_M divX\nu=\int_M d(i(X)\nu)=0.$$

By the same reason as above we have

$$\int_M\triangle f\cdot\nu=\int_M div(gradf)\cdot\nu=\int_M d(i(grad\ f)\nu)=0$$

Q.E.D.

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