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ON QUILLEN'S DEFINITION OF HIGHER K-THEORY

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I. Introduction.

The purpose of this paper is to introduce basic facts on Algebraic K-Theory. As well known, the general definition of higher K given by Quillen [3], is a very noble amalgamation of homotopy theory and category theory. Therefore we start with giving basic results of both fields which we need.

II. Topological background.

The basic topological tool for our purpose is, so called, a simplicial complex and its geometric realization. We will follow, more or less, Milnor [1].

A simplicial complex K conists of a set $\{v\}$ of *vertices* and a set $\{s\}$ of finite nonempty subsets of $\{v\}$ called *simplexes* such that

a) Any set consisting of exactly one vertex is a simplex

b) Any nonempty subset of a simplex is a simplex

A simplex s containing exactly $q+1$ vertices is called a *q-simplex.* If $s'\subset s$ then s' is called a *face* of s.

Then the following facts are obvious:

i) If s is a simplex of a simplicial complex K, the set of all proper faces of s is a simplicial complex denoted by $\dot{s}$.

ii) If K is a simplicial complex, its *q-dimensional skeleton* K^q is defined to be the simplicial complex consisting of all p-simplexes of K for $p\leq q$

iii) Given a set X and a collection $\mathfrak{W}=\{W\}$ of subsets of X, the *nerve* of $\mathfrak{W}$, denoted by $K(\mathfrak{W})$, is the simplicial complex whose simplexes are finite nonempty subsets of $\mathfrak{W}$ with nonempty intersection. Thus the vertices of $K(\mathfrak{W})$ are the nonempty elements of $\mathfrak{W}$.

iv) Let I be a partially ordered set. Then there is a simplicial complex whose set of vertices is I and whose simplexes are finite nonempty totally ordered subset $\{i_0, i_1, \ldots, i_q\}$ such that $i_0\leq i_1\leq\ldots\leq i_q$.

we can define obvious definitions of subcomplex and simplicial maps be-

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tween simplicial complexes.

Furthermore, given $A \subset X$, $\mathfrak{W}=\{W\}$ a collection of subsets of a set X, and let $K_A(\mathfrak{W})$ be the collection of finite nonempty subsets of $\mathfrak{W}$ whose intersection meets A in a nonempty subset, then $K_A(\mathfrak{W})$ is a subcomplex of the nerve $K(\mathfrak{W})$.

Now let us give the final material in topology for our purpose, that is, geometric realization of a simplicial complex K, denoted by $|K|$. As a set, $|K|$ is given as follows: $|K|$ is the set of all functions "α" from the set of vertices to the unit interval I such that

i) For any α, $\{v \in K | \alpha(v) \neq 0\}$ is a simplex of K (in particular, $\alpha(v) \neq 0$ for only a finite set of vertices).

ii) For any α, $\sum_{v \in K} \alpha(v)=1$, where $\alpha(v)$ is called the *vth barycentric coordinate of* α.

Now we are giving topology to $|K|$. First we give the metric topology on $|K|$ defined by the following metric.

$$d(\alpha, \beta)=\left[\sum_{v \in K} |\alpha(v)-\beta(v)|^2\right]^{1/2}$$

and it is denoted by $|K|_d$. We define another topology on $|K|$. For $s \in K$, the *closed simplex* $|s|$ is defined by

$$|s| = \{\alpha \in |K| \,|\, \alpha(v) \neq 0 \Rightarrow v \in s\}.$$

Hence if s is a q-simplex, $|s|$ is in one-to-one correspondence with the set $\{x \in \mathbf{R}^{q+1} | 0 \leq x_i \leq 1, \sum x_i = 1\}$. Also it is easy to see that $|s_1|_d \cap |s_2|_d$ is either empty or $|s_1 \cap s_2|_d$. Hence the family $\{|s| \,|\, s$ simplex of $K\}$ satisfies the definition of closed set of topology. This topology is called *coherent topology*. K with the coherent topology is called *geometric realization of* K. Then the following facts are obvious:

i) A function $f: |K| \rightarrow X$, where X is a topological space, is continuous in the coherent topology if any only if $f| |s| : |s| \rightarrow X$ is continuous for every $s \in K$.

ii) $|K|$ is| a normal Hausdorff space.

Our basic result in this respect is the following:

THEOREM. (due to Milnor). *Let* $K \times K'$ *be the cartesian product of two simplicial complexes, that is,*

$$(K \times K')_n = K_n \times K'_n.$$

The projection maps $\rho : K \times K' \rightarrow K$ *and* $\rho' : K \times K' \rightarrow K'$ *induces maps* $|\rho|$ *and* $|\rho'|$ *of the geometric realizations. A map* $\eta : |K \times K'| \rightarrow |K| \times |K'|$ *is defi-*

ned by $\eta=|\rho|\times|\rho'|$. *Then* η *is homoepmorphism.*

It is called the *compatibility theorem.*

Now we must mention something about classifying space. The essence is also due to Milnor [1], [2]. But for our purpose it must be further generalized. It is due to Segal [4].

III. Algebraic-Topological Background

We generalize simplicial complex defined in II in categorical way [4]. Let (Ord) be the category of finite totally ordered set. Then *simplicial complex with values in* category (C) is defined as a contravariant functor $A:(Ord)\to(C)$. Then we could also define geometric realization $\Delta(A)$ as follows: If, for a finite set S, $\Delta(S)$ denotes the standard simplex with S as set of vertices, then $\Delta(A)$ is obtained from the topological sum of all $\Delta(S)\times A(S)$, for all finite ordered sets S, by identifying $(x,\theta^*a)\in\Delta(S)\times A(S)$ with $(\theta_* x, a)\in\Delta(T)\times A(T)$ for all $x\in\Delta(S)$, $a\in\Delta(T)$, and $\theta: S\to T$ in (Ord). Note that $S\to\Delta(S)$, $S\to A(S)$ are covariant and contravariant functors, respectively. Then we have Milnor's compatibility theorem also.

To a category (we mean small category), we can associate a simplicial set NC, which we might call the nerve of (C), by taking the objects of (C) as vertices and p-simplexes as the diagrams in (C) of the form

$$X_0\to X_1\to\ldots\to X_p$$

The i-th face (resp, degeneracy) of this simplex is obtained by deleting the object X_j (resp. replacing X_j by id : $X_i\to X_i$) in obvious way. Then classifying space of (C), denotes by BC, is the geometric realization of NCdefined as above. Then compatibility theorem says that

$$B(C\times C')=BC\times BC'.$$

Let G be a topological group. For G, we could define a category (G) as follows: $\mathrm{ob}(G)=*$ (point), $\mathrm{mor}(G)=G$. $N(G)$ is given by $NG_k=G\times\ldots\times G$ (k times). And $B(G)$ is exactly classifying space of G originally defined by Milnor.

Let X be an object of (C). Using X to denote also the corresponding 0-cell of BC, we have a family of homotopy groups

$$\pi_i(BC,X),\quad i\geq 0,$$

which will be called the *homotopy groups of* (C) *with base point* X, and denoted simply by $\pi_1(C,X)$. We will see that this abstract definition of homotopy group plays fundamental role in our definition of higher K.

Now let us give some basic facts about BC.

i) $B(C\times C')\rightarrow BC\times BC'$ is a homeomorphism.

ii) A natural transformation $\theta: f\rightarrow g$ of functors from (C) to (C') induces a homotopy $BC\times I\rightarrow BC'$ between Bf and Bg.

iii) If a functor f has either a left or a right adjoint, then Bf is a homotopy equivalence.

IV. Algebraic preliminaries.

We give some definitions which are due to Quillen.

Exact categories: Let (M) be an additive category which is embedded as a full sub-category of an abelian category (A), and suppose that (M) is closed under extensions in (A) in the sense that if an object A of (A) has a suboject A such that A' and A/A' are isomorphic to objects of (M), then A is isomorphic to an object of (M).

Let (E) be the class of sequences

$$0\rightarrow M'\xrightarrow{i} M\xrightarrow{j} M''\rightarrow 0$$

in (M) which are exact in the abelian category (A). We call a map in (M) an *admissible monomorphism* (resp. *admissible epimorphism*) if it occurs as the map i (resp. j) of some member of (E). And they are denoted by $M'\rightarrowtail M$ and $M\twoheadrightarrow M''$ respectively.

Then the class (E) has the following properties:

i) Any sequence in (M) isomorphic to a sequence in (E) is in (E). For any M', M'' in (M), the sequence

$$0\rightarrow M'\rightarrow M'\oplus M''\rightarrow M''\rightarrow 0$$

is in (E).

ii) The class of admissible epimorphisms is closed under composition and under base change by arbitrary maps in (M). Dually the class of admissible monomorphisms is closed under composition and under cobase change by arbitrary maps in (M).

iii) Let $M\rightarrow M''$ be a map possessing a kernel in (M). If there exists a map $N\rightarrow M$ in (M) such that $N\rightarrow M\rightarrow M''$ is an admissible epimorphism, then $M\rightarrow M''$ is an admissible epimorphism. Dually for admissible monomorphisms, the above holds.

DEFINITION: An *exact category* is an additive category (M) equipped with a family (E) of sequences of the form $0\rightarrow M'\rightarrow M\rightarrow M''\rightarrow 0$, called the short exact sequences of (M), such that the properties i), ii), iii) hold. An *exact functor* $F: (M)\rightarrow(M')$ between exact categories is an additive functor carrying exact sequences in (M) into exact sequences in (M').

Now we give a fundamental definition. Of course it is defined by Quillen. The category $Q(M)$. Here Q implies Q of Quillen (of course Quillen did not say it himself).

Let (M) be an exact category defined above, we form a new category $Q(M)$ having the same objects as (M) but morphism defined in the following way: Let M and M' be objects in (M) and consider all diagrams

$$\begin{tikzcd} N \arrow[r, tail, "i"] \arrow[d, two heads, "j"'] & M' \\ M & \end{tikzcd}$$

where j is an admissible epimorphism and i is an admissible monomorphism. We say

$$M \overset{j}{\twoheadleftarrow} N \overset{i}{\rightarrowtail} M' \sim M \overset{j}{\twoheadleftarrow} N' \rightarrowtail M'$$

if

$$\begin{tikzcd} M \arrow[d, equal] & N \arrow[l, two heads, "j"'] \arrow[r, tail] \arrow[d, dotted, "\Phi"'] & M' \arrow[d, equal] \\ M & N' \arrow[l, two heads] \arrow[r, tail] & M' \end{tikzcd}$$

there exist isomorphism $\Phi : N \to N'$ with commutative diagram. Then morphism from M to M' in $Q(M)$ is defined by an isomorphism classes of these diagrams. When a morphism from M' to M'' is represented by the diagram $M' \overset{j'}{\twoheadleftarrow} N' \overset{i'}{\rightarrowtail} M''$, then the composition of this morphism with the morphism from M to M' is the morphism represented by the pair $j \cdot pr_1$, $i \cdot pr_2$ in the diagram

$$\begin{tikzcd} N \times N \arrow[r, tail, "pr_2"] \arrow[d, "pr_1"'] & N' \arrow[r, tail, "i'"] \arrow[d, "j'"'] & M'' \\ N \arrow[r, tail, "i"] \arrow[d, "j"'] & M' & \\ M & & \end{tikzcd}$$

Then it is clear that composition is well-defined and associative. And we assume $Q(M)$ is a well defined category (i. e., isomorphism class of diagram

$M \twoheadleftarrow{j} N \rightarrowtail M'$ form a set).

Now we are ready for the definition of higher-K's.

V. Definition of higher K-groups

DEFINITION: The higher K-groups for a small exact category (M) are defined as follows:

$$K_i(M) \underset{\text{def}}{=\!=\!=} \pi_{i+1}(B(Q(M)),0)$$

THEOREM 1: *$\pi_1(B(Q(M)),0)$ is canonically isomorphic to the Grothendieck group $K_0(M)$.*

THEOREM 2: *K_i is a functor from exact categories and exact functors to abelian groups.*

EXAMPLE 1: Let A be a ring with 1 and $P(A)$ denote the additive category of finitely generated projective A-modules. We may regard $P(A)$ as an exact category. We define $K_i(A)=K_i(P(A))$. Then this is the usual definition of higher K-groups of a ring A.

EXAMPLE 2: If X is a scheme, we put $K_q(X)=K_q(P(X))$, where $P(X)$ is the category of vector bundles over X equipped with the usual notion of exact sequence.

THEOREM 3. *If A is the ring of algebraic integers in a number field F (finite over $\mathbf{Q}$), then K_iA is a finitely generated group for all $i>0$.*

THEOREM 4: *Let A be a Dedekind domain with field of fractions F. We have an exact sequence*

$$K_{n+1}F \longrightarrow \oplus_m K_n(A/m) \longrightarrow K_nA \longrightarrow K_nF \longrightarrow \cdots$$

where m runs over the nonzero maximal ideals of A.

THEOREM 5. *Let $\mathbf{F}_q$ be the finite field of q elements.*

Then $K_0\mathbf{F}_q=\mathbf{Z}$, $K_{2i}\mathbf{F}_q=0$ for all $i\geq 1$. $K_{2i-1}\mathbf{F}_q$ is cyclic of order q^i-1 for $i\geq 1$.

THEOREM 6. *Let A be the ring of integers in a number field having r_1 real and r_2 complex places. Then the dimension of $K_nA\otimes\mathbf{Q}$ is*

1 *for* $n=0$, r_2+r_1-1 *if* $n=1$. *And if* $n\geq 2$ *it is* 0, r_1+r_2, 0, r_2 *as* $n\equiv 0,1,2,3 \pmod 4$, *respectively.*

THEOREM 7. *Let* $(\mathscr{P})$ *be a full subcategory of* $(\mathscr{M})$ *such that*

(i) *for any exact sequence in* $(\mathscr{M})$,

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we have

a) $M', M'' \in (\mathscr{P}) \Rightarrow M \in (\mathscr{P})$

b) $M \in \mathscr{P} \Rightarrow M' \in \mathscr{P}$

(ii) *for every object* M'' *of* $(\mathscr{M})$, *there exists an exact sequence* $0 \to M' \to M \to M'' \to 0$ *with* M *and* M' *in* M''.

Then the indurced map $|Q(\mathscr{P})| \to |Q(\mathscr{M})|$ *is a homotopy equivalence. Especially* $K_i((\mathscr{P})) = K_i((\mathscr{M}))$ *for every* $i \geq 0$.

THEOREM 8. *Let* $\mathscr{G}$ *be a finite dimensional Lie algebra over a field* k *and* $\mathscr{U}(\mathscr{G})$ *its enveloping algebra. Then* $K_i(k) \cong K_i(\mathscr{U}(\mathscr{G}))$ *for every* $i \geq 0$.

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