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DETERMINANTS OF n-DIMENSIONAL MATRICES

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1. Introduction

We denote by $M_n(p)$ the set of all $p\times p\times\cdots\times p=p^n$ matrices over the real numbers. Any matrix A in M_n is called an n-dimensional matrix. If A is a square matrix or an $p\times p$ matrix, then A is a 2-dimensional matrix. There are many kinds of determinants (see [1] and [2] for definitions of determinants) for n-dimensional matrices (where n is a positive integer greater than 2).

We first define a determinant $\det_j(A)$ of an n-dimensional matrix A (for $j=1, 2, \ldots, n$) following the (Japanese language) paper [1] by R. Kaneiwa. We also define a product AB of two $2m$-dimensional matrices A and B in $M_{2m}(p)$. We prove that the associative law holds for the product AB in $M_{2m}(p)$. We obtain the identity matrix of $M_{2m}(p)$. We study elementary properties of $\det_j(A)$ for $A\in M_n(p)$, prove that $\det_j(AB)\neq\det_j(A)\det_j(B)$ and compute determinants of identity matrices as $\det_j(I)=(p!)^{m-1}$, where I is the identity matrix in $M_{2m}(p)$.

2. Definition of a determinant

In this section we give a definition of a determinant $\det_j(A)$ of an n-dimensional matrix A in $M_n(p)$, for $j=1, 2, \ldots, n$. We assume all matrices are real matrices. Let p be a positive integer greater than 1. $S(p)$ denotes the symmetric group on the set $\{1, 2, \ldots, n\}$. A matrix A is called an n-dimensional (square) matrix over $S(p)$ if A is an $p\times p\times\cdots\times p=p^n$ matrix. We denote by $M_n(p)$ the set of all n-dimensional matrices over $S(p)$. (A 2-dimensional square matrix A over $S(p)$ is a $p\times p$ matrix and we call it just a square matrix). We define a set $S^n(p)=\{\lambda=(\lambda_1\lambda_2\cdots\lambda_n) : \lambda_i\in S(p)\}$. We define $\pi\lambda$ for $\pi\in S(p)$ and $\lambda=(\lambda_1\lambda_2\cdots\lambda_n)\in S^n(p)$ by $\pi\lambda=(\pi\lambda_1\,\pi\lambda_2\cdots\pi\lambda_n)$. We define a relation R on $S^n(p)$ by $\lambda R\,\mu$ iff $\mu=\pi\lambda$ (for some $\pi\in S(p)$), for $\lambda, \mu\in S^n(p)$. It is clear that R is an equivalence relation on $S^n(p)$ and we define $S^n(p)/R\equiv S_n(p)$ as the set of all equivalence classes defined by R. Let $A\in M_n(p)$. An entry of A takes the form $a_{i_1i_2\cdots i_n}$. Let

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$\pi=\begin{pmatrix}1 & 2 \cdots p \\ 1' & 2' \cdots p'\end{pmatrix} \in S(p)$. Then we can write $\pi(i)=i'$. Letting $\lambda=(\lambda_1\lambda_2\cdots\lambda_n)\in S^n(p)$, define

$$a_\lambda=\prod_{i=1}^{p} a_{\lambda(i)}=a_{\lambda_1(1)\lambda_2(1)\cdots\lambda_n(1)}a_{\lambda_1(2)\lambda_2(2)\cdots\lambda_n(2)}\cdots a_{\lambda_1(p)\lambda_2(p)\cdots\lambda_n(p)},$$

as a product of p entries of the matrix A.

LEMMA 1. *Let* $[\lambda]\in S_n(p)$ *and let* $A\in M_n(p)$. *Then* $a_\lambda=a_\mu$, *for* $\mu\in[\lambda]$.

REMARK $a_\lambda=a_\mu$ means that a_λ and a_μ are identical when we apply the commutativity of the real numbers.

Proof. We prove it by induction on n. If $n=1$, it is trivial. Assume that we have proved it for $n<k$, where k is a fixed positive integer greater than 1. Let $n=k$ and $\mu=(\mu_1\,\mu_2\ldots\mu_k)=\pi\,\lambda=(\pi\,\lambda_1\;\pi\,\lambda_2\ldots\pi\,\lambda_k)$ for $\mu\in[\lambda]$. Without loss of generality we can assume that $\pi=(1\;i)$ (a transposition and $i\neq 1$) and $\lambda_1=I$, the identity of the group $S(p)$. We consider $(\pi\lambda_1(1)\;\pi\,\lambda_2(1)\ldots\pi\,\lambda_k(1))=K_1$. We see that $\pi\,\lambda_1(1)=i$ and by inductional assumption we have that $K_1=(i\,\lambda_2(t)\lambda_3(t)\ldots\lambda_k(t))$ for some t. We have $\pi\,\lambda_j(1)=\lambda_j(t)$ $(j=2,3,\ldots k)$ from which we get that $\lambda_j(i)=\lambda_j(t)$ and hence we have $a_{\lambda(i)}=a_{\lambda_1(i)\lambda_2(i)\cdots\lambda_k(i)}=a_{\mu(1)}$. Now consider $K_2=(\pi\lambda_1(i)\;\pi\lambda_2(i)\ldots\pi\lambda_k(i))=(1\;\pi\lambda_2(i)\;\pi\lambda_3(i)\ldots\pi\lambda_k(i))$. By inductional assumption we can have that $K_2=(1\;\lambda_2(t)\;\lambda_3(t)\ldots\lambda_k(t))$ from which we obtain that $\pi\lambda_j(i)=\lambda_j(t)$ and $\pi\lambda_j(i)=\lambda_j(1)=\lambda_j(t)$ and hence we have that $K_2=(\lambda_1(1)\;\lambda_2(1)\ldots\lambda_k(1))$. Therefore we have $a_{\lambda(1)}=a_{\mu(i)}$. Finally we consider $K_3=(\pi\lambda_1(j)\;\pi\lambda_2(j)\ldots\pi\lambda_k(j))$ for $1\neq j\neq i$. We can show that $K_3=(\lambda_1(j)\;\lambda_2(j)\ldots\lambda_k(j))$ and hence $a_{\lambda(j)}=a_{\mu(j)}$ and $a_\lambda=a_\mu$. This proves the lemma.

By Lemma 1, we define $a_{(\lambda)}$ as $a_{(\lambda)}=a_\mu$ for $\mu\in[\lambda]$. Now we define sign $_j([\lambda])$ for $[\lambda]\in S_n(p)$. For j there exists $u=(u_1\,u_2\ldots u_n)\in[\lambda]$ such that $u_j=I$, the identity of the group $S(p)$. $\mathrm{sign}_j([\lambda])=\prod_{i=1}^{n}(\mathrm{sign}(u_i))$ is defined as the product of all $\mathrm{sign}(u_i)$.

We have now a definition of a determinant of A.

DEFINITION 1. Let $A=(a_{ij\ldots k})\in M_n(p)$ be an n-dimensional matrix over $S(p)$ and A is a real matrix. We define

$$\det_j(A)=\sum_{[\lambda]\in S_n(p)}\mathrm{sign}\,([\lambda])\,a_{[\lambda]}$$

the summation being taken for all elements $[\lambda]$ in $S_n(p)$. We may call $\det_j(A)$ a j-determinant of an n-dimensional matrix.

Note that if A is a $p\times p$ matrix, then $\det_1(a)=\det_2(A)=\det(A)$. $|S|$ denotes the cardinality of a set S.

LEMMA 2. *Let A be as in Definition 1. Then $det_j(A)$ has $(p!)^{n-1}$ terms in its expansion, $j=1, 2, \ldots, n$.*

Proof. We can see that $|S_n(p)|=(p!)^{n-1}$.

3. Elementary properties of determinants

In this section we shall prove that if n is even, then $\det_j(A)=\det_1(A)$ for all $j=2, 3, \ldots, n$. In the case $n=2m$, we just write $\det(A)$ instead of $\det_j(A)$. We first construct an example.

EXAMPLE 1. Let A be a 3-dimensional matrix over $S(3)$ and let

$$A=\begin{bmatrix} a_{111} & a_{112} & a_{113} & a_{211} & a_{212} & a_{213} & a_{311} & a_{312} & a_{313} \\ a_{123} & a_{122} & a_{123} & a_{221} & a_{222} & a_{223} & a_{321} & a_{322} & a_{323} \\ a_{131} & a_{132} & a_{133} & a_{231} & a_{232} & a_{233} & a_{331} & a_{332} & a_{333} \end{bmatrix}$$

with $a_{112}=1$, $a_{123}=4$, $a_{122}=7$, $a_{211}=6$, $a_{212}=5$, a_{221} and $a_{333}=3$, and all other entries of A are zero. Let $[\lambda]=[(I\,I\,(12)]$, $[u]=[(I\,(12)\,I)]$ and $[v]=[((12)\,I\,I)]$. Then we see that

$$\mathrm{sign}_1[\lambda]=\mathrm{sign}_2[\lambda]=\mathrm{sign}_1[u]=\mathrm{sign}_3[u]=\mathrm{sign}_2[v]=\mathrm{sign}_3[v]=-,$$
$$\mathrm{sign}_3[\lambda]=\mathrm{sign}_2[u]=\mathrm{sign}_1[v]=+,\ a_{[\lambda]}=6,\ a_{[u]}=60 \text{ and } a_{[v]}=42.$$

We can see that $a_{[\lambda]}$, $a_{[u]}$ and $a_{[v]}$ are only non-zero terms of the expansion of each $\det_j A$, $\det_1 A=-24$, $\det_2 A=12$ and $\det_3 A=-96$.

THEOREM 1. *Let $n=2m$ $(m\geqq 2)$ and let $A=(a_{ij\ldots k})\in M_n(p)$. Then $\det_1(A)=\det_j(A)$ for $j=2, 3, \ldots, n$.*

Proof. Let $[\lambda]$ be an arbitrary memberof $S_n(p)$ and consider $\mathrm{sign}_1([\lambda])$. Without loss of generality we can assume that $\mathrm{sign}_1([\lambda])=+$, $\lambda=(\lambda_1\,\lambda_2\ldots\lambda_n)$ and $\lambda_1=I$. We suppose that $\mathrm{sign}_j([\lambda])=-$ $(j\neq 1)$, $\pi\lambda=u=(u_1\,u_2\ldots u_n)=(\pi\lambda_1\,\pi\lambda_2\ldots\pi\lambda_n)$ and $u_j=I$. Note that $\mathrm{sign}_1([\lambda])=\mathrm{sign}(\lambda)=\prod_{i=1}^{n}(\mathrm{sign}\ \lambda_i)$ and $\mathrm{sign}_j([\lambda])=\mathrm{sign}(u)=-$. From $\mathrm{sign}(u)=-$, there are $u_{t(1)}, u_{t(2)}, \ldots, u_{t(2q+1)}$ such that $\mathrm{sign}(u_{t(i)})=-$ and $\mathrm{sign}(u_s)=+$, for $s\neq t(i)$ $(i=1, 2, \ldots, 2q+1)$. We can see that $+=\mathrm{sign}(\lambda)=\prod_{i=1}^{n}(\mathrm{sign}\ \lambda_i)=\prod_{i=1}^{n}(\mathrm{sign}(\pi^{-1}u_i))=\prod(\mathrm{sign}(u_i))=\mathrm{sign}(u)=-$, a contradiction. This proves the theorem.

THEOREM 2. *Let $A=(a_{ij\ldots k})\in M_{2m+1}(p)$. Then there are $2m+1$ distinct determinants $\det_j(A)$, $j=1, 2, \ldots, 2m+1$.*

Proof. For $n=3$, see Example 1. Let $n=2m+1$ $(m\geqq 2)$. We need notations. Let $V_n(p)=\{(i_1\,i_2\ldots i_n) : i_j$ is a positive integer such that $1\leqq i_j\leqq p\}$ and d_{ij} denotes the Krojecker's delta $(d_{ij}=1$ if $i=j$ and $d_{ij}=0$ if $i\neq j)$. Let $d_i=(d_{i1}d_{i2}\ldots d_{in})\in V_n(p)$. In $V_n(p)$, define $e(1)=d_1+d_2+\cdots+d_n=(11\ldots 1)$,

$e(i)=ie(1)=(i\,i\ldots i)$, $e(1\,i)=e(1)+d_1$ and $e(2\,i)=e(2)+d_i$. With these vectors we define all non-zero entries of A as follows. $a_{e(i)}=1$ for all $i\geqq 3$, $a_{e(1\,n)}=a_{e(2\,n)}=1$, $a_{e(1\,n-1)}=a_{e(2\,n-1)}=\sqrt{2}$, ..., $a_{e(1\,n-i+1)}=a_{e(2\,n-i+1)}=\sqrt{i}$, ..., $a_{e(1\,1)}=a_{e(2\,2)}=\sqrt{n}$ (and all other entries of A are equal to 0). We now define $\lambda(i)=(\lambda_1\,\lambda_2\ldots\lambda_n)\in S^n(p)$ as follows: $\lambda_{n-i+1}=(1\,2)$, a transposition, and $\lambda_t=I$ (the identity) for $t\neq n-i+1$. Then we can see that

$$\mathrm{sign}_j([\lambda(i)])=\begin{cases}+ & \text{if } i=n-j+1,\\ - & \text{otherwise.}\end{cases}$$

With these data we can compute $\det_j(A)$ and obtain that $\det_j(A)=-(n+1)n/2+2(n-j+1)$. We can check that all $\det_j(A)$ are distinct for $j=1,2,\ldots,n=2m+1$. This proves the theorem.

We shall establish a theorem which is analogous to that if any two rows of a matrix A are identical then $\det A=0$ for a 2-dimensional matrix A. To do this we introduce notations. We recall that $V_n(p)=\{(i_1\,i_2\ldots i_n):i_j$ are integers such that $1\leqq i_j\leqq p\}$. Letting $\lambda\in V_n(p)$, a_λ denotes an entry of $A=(a_{ij\ldots k})\in M_n(p)$. (Note that we have used a_λ $(\lambda\in S^n(p))$ as a product of p entries of A in the section 2). We define $\lambda(i\,j)=(\lambda_1\,\lambda_2\ldots\lambda_n)\in V_n(p)$ by $\lambda_i=j$. Let $A=(a_{ij\ldots k})\in M_n(p)$. Define $A^m{}_i=(a_{\lambda(m\,i)})$ as a submatrix of A, and we call $A^m{}_i$ the ith row (or face) of A in the m-direction. For simplicity, we often denote $A^1{}_i$ by A_i, and we may call A_i the ith row (or face) of A.

THEOREM 3. *Let* $A=(a_{ij\ldots k})\in M_n(p)$.

(1) *Let* $B=(b_{ij\ldots k})\in M_n(p)$ *be the matrix obtained from* A *by multiplying row* i_0 *of* A *by scalar* r *(that is,* $B_i=A_i$ $(i\neq i_0)$ *and* $B_{i_0}=rA_{i_0}$*). Then*
$$\det_j B=r\det_j A.$$

(2) *Let* B *be obtained from* A *by interchanging the ith row and the kth row of* A *(that is,* $B_i=A_k$, $B_k=A_i$ $(i\neq k)$ *and* $B_t=A_t$ $(i\neq t\neq k)$ *for* $B=(B_1B_2\ldots B_p)$*). Then* $\det_1 B=-\det_1 A$.

Proof. We omit the proof of (1) and we consider (2). Let $[\lambda]$ be an arbitrary member of $S_n(p)$ and without loss of generality we can assume that $\lambda_1=1$ for $\lambda=(\lambda_1\,\lambda_2\ldots\lambda_n)$. Define $u=(u_1\,u_2\ldots u_n)$ by $u_1=(i\,k)$ and $u_t=\lambda_t$ $(t\neq 1)$. Then $\mathrm{sign}_1([\lambda])=-\mathrm{sign}_1([u])$ and we can show that

$$\det_1 B=\sum_{[u]\in S_n(p)}\mathrm{sign}_1([u])\,b_{[u]}=-\sum_{[\lambda]\in S_n(p)}\mathrm{sign}_1([\lambda])\,a_{[\lambda]}=-\det_1 A$$

This proves the theorem.

4. A Product of two matrices

For $A=(a_{ij\ldots k})$, $B=(b_{ij\ldots k})\in M_{2m}(p)$, we define a product $AB=C=$

$(c_{ij\cdots k})$ of two matrices A and B as follows:

$$c_{i_1i_2\cdots i_{2m}}=\sum_{t^i=1}^{p}(a_{i_1i_2\cdots i_mt_1t_2\cdots t_m})(b_{t_1t_2\cdots t_mi_{m+1}i_{m+2}\cdots i_{2m}}).$$

We can see that $C\in M_{2m}(p)$.

LEMMA 3. *Let $M_{2m}(p)$ be the set of all $2m$-dimensional matrices over $S(p)$. Then $(AB)C=A(BC)$ for all A, B, C in $M_{2m}(p)$. Thus $M_{2m}(p)$ forms a semigroup under the matrix product defined in the above.*

We omit the proof of the lemma. We define a matrix.

DEFINITION 2. Let $\lambda\in V_m(p)$. For λ we use a notation $\lambda\lambda=(\lambda\ \lambda)\in V_{2m}(p)$ as a vector with $2m$ components. Let $B=(b_{ij\cdots k})$ be a matrix in $M_{2m}(p)$ defined as follows: $b_{\lambda\lambda}=1$ for $\lambda\in V_m(p)$ and all other $b_u=0$ $(u\in V_{2m}(p)$ and $u\neq\lambda\lambda)$. We denote this B by I and we call it the identity matrix of the semigroup $M_{2m}(p)$.

LEMMA 4. *Let I be a matrix defined in Definition 2. Then*

$$IA=AI=A \text{ for all } A\in M_{2m}(p).$$

Proof. Let $I=(b_{ij\cdots k})$, $A=(a_{ij\cdots k})$ and $AI=C=(c_{ij\cdots k})$. Then

$$\begin{aligned}c_{i_1i_2\cdots i_{2m}}&=\sum_{t^i}(a_{i_1i_2\cdots i_mt_1t_2\cdots t_m}\ b_{t_1t_2\cdots t_mi_{m+1}i_{m+2}\cdots i_{2m}})\\&=a_{i_1i_2\cdots i_mi_{m+1}\cdots i_{2m}}\ b_{i_{m+1}i_{m+2}\cdots i_{2m}i_{m+1}i_{m+2}\cdots i_{2m}}\\&=a_{i_1i_2\cdots i_mi_{m+1}i_{m+2}\cdots i_{2m}},\end{aligned}$$

since $b_{\lambda\lambda}=1$ and $b_{uv}=0(u\neq v)$. We can prove that $IA=A$. This proves the lemma.

Combining Lemmas 3 and 4 we have the following.

THEOREM 4. *$M_{2m}(p)$ is a semigroup with the identity I.*

THEOREM 5. *Let $A, B\in M_{2m}(p)$. Then* $\det(AB)\neq\det A\ \det B$.

Proof. Let I be the identity of $M_4(3)$. Then we can compute that $\det I=6$. This proves the theorem.

5. Determinants of identity matrices

We shall prove the following theorem.

THEOREM 6. *Let I be the identity matrix of the semigroup $M_{2m}(p)$. Then*

$$\det(I)=(p!)^{m-1}.$$

REMARK. In the proof of Theorem 5, we mentioned that, for $I\in M_4(3)$, $\det I=3\ =6$, which is a part of Theorem 6. For the identity matrix I in $M_2(p)$, we know that $\det I=1$, which is also a part of Theorem 6.

Proof. Define $V_1=\{(\lambda_1\lambda_2\dots\lambda_m)\lambda\in V_m(p) : \lambda_1=1\}$. Similarly, we define $V_i=\{(\lambda_1\lambda_2\dots\lambda_m)=\lambda\in V_m(p) : \lambda_1=i\}$. Let $I=(a_{ij\dots k})$ be the identity matrix of the semigroup $M_{2m}(p)$. Then any non-zero entry of I is of the form $a_{\lambda\lambda}$ $(\lambda\in V_m(p))$. Define $E(I)=\{a_{\lambda\lambda} : \lambda\in V_m(p)\}$ as the set of all non-zero entries of I. Note that $a_{\lambda\lambda}=1$. Define $I_1=\{a_{\lambda\lambda}\in E(I) : \lambda\in V_1\}$. Then we can see that $|I_1|=p^{m-1}$. We recall that $e(1)=(1\,1\dots 1)\in V_m(p)$. Let $B=\bar{a}_1\bar{a}_2\dots\bar{a}_p$ be a term of the expansion of the $\det(I)$. We can pick $\bar{a}_1$ from I_1 and we can assume that $\bar{a}_1=a_{e(1)e(1)}$. For $a_{e(1)\,e(1)}$, we define $U_2=\{\lambda=(2\,\lambda_2\,\lambda_3\dots\lambda_m)\in V_2 : \lambda_i\geqq 2\}$ and define $I_2=\{a_{\lambda\lambda}\in E(I) : \lambda\in U_2\}$. We can see that $|I_2|=(p-1)^{m-1}$. We can see that $\bar{a}_2$ must be a member of I_2. We can assume (without loss of generality) that $\bar{a}_2=a_{e(2)e(2)}$, where $e(2)=2e(1)=(2\;2\dots 2)\in V_m(p)$. For $B=a_{e(1)e(1)}a_{e(2)e(2)}\bar{a}_3\dots\bar{a}_p$, we define $U_3=\{\lambda=(\lambda_1\lambda_2\dots\lambda_m)\in V_3 : \lambda_i\geqq 3,\ i\neq 1\}$ and define $I_3=\{a_{\lambda\lambda} : \lambda\in U_3\}$. Note that $|I_3|=(p-2)^{m-1}$. We see that $\bar{a}_3$ belongs to I_3. Inductively, for $\bar{a}_i=a_{e(i)e(i)}$, we define $U_{i+1}=\{\lambda=(\lambda_1\lambda_2\dots\lambda_m)\in V_{i+1} : \lambda_i\geqq i+1\}$ and define $I_{i+1}=\{a_{\lambda\lambda}\in E(I) : \lambda\in U_{i+1}\}$. Then we can show that $|I_{i+1}|=(p-i)^{m-1}$ and $\bar{a}_{i+1}$ must be a member of I_{i+1}. Therefore we can say that the total number of such terms $B=\bar{a}_1\,\bar{a}_2\dots\bar{a}_p$ in the expansion of the determinant of I is equal to $(p!)^{m-1}$ because of that every term B takes the $+$ sign, that is, $B=1$. This proves the theorem.

PROBLEM. Prove or disprove that $\det(AB)=c(\det(A))(\det(B))$, where c is a constant and $A, B\in M_{2m}(p)$.

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