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REDUCING OPERATOR VALUED SPECTRA OF A HILBERT SPACE OPERATOR

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1. Introduction

The study of reducing operator valued spectra of a Hilbert space operator was initiated by D. W. Hadwin in ([4], [5]), and further progresses have been made in his subsequent papers ([6], [7], [8]). Our paper is regarded as a continuation of [9] which was an attempt to add still other information to the Hadwin's works above. Throughout, H denotes a separable infinite dimensional Hilbert space over the complex numbers, $\mathcal{B}(H)$ the set of all bounded linear transformations on H, and $\mathcal{K}(H)$ the ideal of compact operators in $\mathcal{B}(H)$. For brevity, a bounded linear transformation is called an operator. Also a closed linear submanifold M of H will be called a subspace of H and denoted by $M \leqq H$.

2. Reducing eigenspaces

We first note that the original definition ([5], p. 332) of reducing eigenspace $\mathrm{Eig}(A; T)$ of an irreducible operator $A \in \mathcal{B}(M)$ for $T \in \mathcal{B}(H)$ was extended ([9], p. 131. Definition 1) by removing the irreducibility constraint for A. Thus $f \in \mathrm{Eig}(A; T)$ if and only if $p_n(T, T^*) f \rightarrow 0$ weakly in H, whenever $\{p_n(x, y)\}$ is a sequence of noncommutative polynomials such that $p_n(A, A^*) \rightarrow 0$ in the weak operator topology in $\mathcal{B}(M)$. With this extension, $\mathrm{Eig}(A; T)$ is still a (closed) reducing subspace of T ([9], p. 132, Proposition 1). Professor D. W. Hadwin commented that Proposition 1 of [9] can be also proven by using facts in [2]. We give a characterization of $\mathrm{Eig}(A; T)$ in the next theorem. The notation $\simeq$ means the unitary equivalence. A *suboperator* A_1 of A is the restriction of A to a nonzero reducing subspace M_1 of A. When an operator B is unitarily equivalent to a suboperator of A, we shall write $B \leq A$. Two operators A and T are called *disjoint*, denoted by $A \,\delta\, T$ if no suboperator of A is unitarily equivalent to any suboperator of T.

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THEOREM 1. *Let M and N be two nonzero separable Hilbert spaces. Assume that $A\in\mathcal{B}(M)$ and $T\in\mathcal{B}(N)$. Then*

(i) $\mathrm{Eig}(A; T)=\bigvee\{K\leqq N; K\neq \{0\}$, *$K$ reduces T and $T|K\leq A\}$, provided* $\mathrm{Eig}(A; T)\neq\{0\}$.

(ii) *$A ⫰ T$ if and only if* $\mathrm{Eig}(A; T)=\{0\}$.

Proof. (i) Assume that Eig $(A; T)\neq\{0\}$. Let $K\leqq N$, $K\neq\{0\}$, K reduces T and $T|K\simeq A_1\leq A$. Assume that $\{p_n(x, y)\}$ be a sequence of noncommutative polynomials such that $p_n(A, A^*)\to 0$ in the weak operator topology in $\mathcal{B}(M)$. Thus, $p_n(A_1, A_1^*)\to 0$ in the weak operator topology in $\mathcal{B}(M_1)$, where $M_1\leqq M$. It follows that $p_n(T|K, (T|K)^*)\to 0$ in the weak operator topology in $\mathcal{B}(K)$, which, in turn, implies that $p_n(T|K, (T|K)^*)f\to 0$ weakly in K, for each $f\in K$. But K reduces T, so $p_n(T, T^*)f\to 0$ weakly in N, for each $f\in K$. Hence $f\in\mathrm{Eig}(A; T)$, showing $K\subset\mathrm{Eig}(A; T)$. It follows that $\mathrm{Eig}(A ; T)$ contains the right hand side, say N_2, of (i). Write $N_1=\mathrm{Eig}(A ; T)\ominus N_2$ and assume that $N_1\neq\{0\}$. By the assertion (2) of ([9], p. 132), we find D and A_1 such that $D\leq T|N_1$, $A_1\leq A$ and $D\simeq A_1\leq A$. The space N_3 on which D acts also reduces T and $D=T|N_3$. Now by definition of N_2, we see that $N_3\subset N_2$, which is a contradiction to the fact that $N_3\subset N_1$. Hence $N_1=\{0\}$, that is, $\mathrm{Eig}(A ; T)=N_2$.

(ii). ($\Rightarrow$) This is clear from (i).

($\Leftarrow$) Suppose that $A ⫰ T$. It suffices to show that $\mathrm{Eig}(A ; T)\neq\{0\}$. Let $A_1\leq A$, $A_1\simeq T_1\leq T$, and N_1 be the space on which T_1 acts. By using the definition of $\mathrm{Eig}(A ; T)$, it is immediate to check that $\{0\}\subset N_1\subset\mathrm{Eig}(A ; T)$. Q. E. D.

3. Spectral properties of T pertaining to $C^*(T)$ and $W^*(T)$

Two operators $S\in\mathcal{B}(L)$ and $T\in\mathcal{B}(H)$ are called *approximate equivalent*, denoted by $S \underline{a} T$, if there is a sequence $\{U_n\}$ of unitary operators such that $U^*_nTU_n-S$ is compact and $\|U^*_nTU_n-S\|\to 0$ ([5], p. 330, Definition 1. 1). It is known that there are a few interesting characterizations of the approximate equivalence, when L and H are infinite dimensional separable Hilbert spaces ([5], p. 337, Theorem 3. 5, Corollary 3. 6 and Corollary 3. 7). In the following theorem we will give another characterization of the approximate equivalence. For a subset $\mathcal{S}$ of $\mathcal{B}(H)$, the *effective subspace* for $\mathcal{S}$ in H is the (closed) subspace of H spanned by $\{S(H) ; S\in\mathcal{S}\}$. If $T\in\mathcal{B}(H)$. Then $C^*(T)$ (resp. $W^*(T)$) be the C^*-algebra (resp. W^*-algebra) in $\mathcal{B}(H)$ generated by T and T

THEOREM 2. *Let L and H be infinite dimensional separable Hilbert spaces* *and* $S \in \mathcal{B}(L)$ *and* $T \in \mathcal{B}(H)$. *Then,* a T *if and only if*

(i) *there exists a* $*$*-isomorphism* Π *from* $C^*(S)$ *onto* $C^*(T)$ *such that* $\Pi(S)=T$ *and* $\Pi(I)=I$,

(ii) Π *maps* $C^*(S) \cap \mathcal{K}(L)$ *onto* $C^*(T) \cap \mathcal{K}(H)$, *and*

(iii) $C^*(S) \cap \mathcal{K}(L)$ *and* $C^*(T) \cap \mathcal{K}(H)$ *are unitarily equivalent through* Π, *when they are restricted to the corresponding effective subspaces.*

Proof. ($\Rightarrow$) By Corollary 3.7 ([5], p. 337), there is a $*$-isomorphism $\Pi : C^*(S) \to C^*(T)$ such that $\Pi(S)=T$, $\Pi(I)=I$ and $\operatorname{rank} A = \operatorname{rank} \Pi(A)$, for every $A \in C^*(S)$. Then (i) and (ii) are clear, by noticing that any compact operator in a C^*-algebra can be approximated in norm by a sequence of finite rank operators in the C^*-agebra. To see (iii), let L_e and H_e be the effective subspaces for $C^*(S) \cap \mathcal{K}(L)$ and $C^*(T) \cap \mathcal{K}(H)$ respectively. Regard Π as a representation of $C^*(S)$ into $\mathcal{B}(H)$. Let σ be the inclusion representation of $C^*(S)$ into $\mathcal{B}(L)$. Then the two representation Π and σ of $C^*(S)$ are approximately equivalent ([5], p. 330, Definition 1.1). By Theorem 5 ([1], p. 342), we see that the following hold;

(a) Π is a $*$-isomorphism,

(b) $C^*(\mathrm{T}) \cap \mathcal{K}(H) = \{A \in C^*(S) \ ; \Pi(A) \in \mathcal{K}(H)\}$

and

(c) the restrictions $\Pi C^*(S)$ are unitarily equivalent when they are regarded as representations into $\mathcal{B}(H_e)$ and $\mathcal{B}(L_e)$ respectively. Here recall that H_e (resp. L_e) is invariant under $C^*(T) \cap \mathcal{K}(H)$ (resp. $C^*(S) \cap \mathcal{K}(L)$). Again (a) and (b) are (i) and (ii) respectively. Now (b) and (c) imply that

$(C^*(T) \cap \mathcal{K}(H)) \,|\, H_e \simeq (C^*(S) \cap \mathcal{K}(L)) \,|\, L_e.$

($\Leftarrow$) If $A \in C^*(S)$ and rank $A < \infty$. Then $A \in C^*(S) \cap \mathcal{K}(L)$. Then rank A $= \operatorname{rank} \Pi(A)$, by (ii) and (iii). Consequently, ran k$A = \operatorname{rank} \Pi(A)$, for every $A \in C^*(S) \cap \mathcal{K}(L)$. For an operator $A \in C^*(S) \cap \mathcal{K}(L)$, we note that rank $A = \infty$. By the similar argument, we see that $\operatorname{rank} \Pi(A) = \infty$. Hence $\operatorname{rank}(A) = \operatorname{rank} \Pi(A)$, for every $A \in C^*(S)$. By Corollary 3.7 ([5], p. 337), we see that $S \underset{a}{\sim} T$. Q. E. D.

We say that an operator S *covers* an operator T (denoted $S \} T$) if no suboperator of T is disjoint from S. S is *quasiequivalent* to T (denoted $S \approx T$) if S covers T and T covers S ([3], p. 9, Definition 1.10). It is proven in Proposition 1.11 ([3], p. 10) that $S \approx T$ if and only if $\infty S \simeq \infty T$, where ∞S denotes the direct sum of infinite number of copies of S, and this, in turn, is equivalent to the fact that there is a $*$-isomorphism Π on $W^*(S)$ onto $W^*(T)$ such that $\Pi(S)=T$ and $\Pi(I) = I$ ([3], p. 19). The next Theorem 3 is an analogue to Theorem 2.8 ([5], p. 334). (i) and (ii) in the theorem was stated in ([9], p. 135) without proof.

THEOREM 3. *Let $A\in\mathcal{B}(M)$ and $T\in\mathcal{B}(H)$, where H is an infinite dimensional separable space. Then the following conditions are equivalent.*

(i) $A\in\Sigma_0(T)$ *and* $\mathrm{mult}(A\,;\,T)<\infty$.

(ii) *There is an irreducible C^*-representation Π of $W^*(T)$ into $\mathcal{B}(M)$ such that $\Pi(T)=A$, $\Pi(I)=I$ and $W^*(T)\cap\mathcal{K}(H)\not\subset\ker(\Pi)$.*

(iii) *A is an irreducible operator and it is unitarily equivalent to a suboperator $T_1\in\mathcal{B}(H_1)$ of T, where H_1 is contained in the effective subspace of $W^*(T)\cap\mathcal{K}(H)$.*

Proof. (i)$\Rightarrow$(ii).
Indeed, let $A\in\Sigma_0(T)$ and $\mathrm{mult}(A\,;\,T)<\infty$. Then $T=T_1\oplus T_2\oplus\ldots\oplus T_n\oplus S$, where $n=\mathrm{mult}\,(A\,;\,T)<\infty$, $T_1\simeq T_2\simeq\ldots\simeq T_n\simeq A$ and $A\,\underset{\smile}{\mid}\,S$. Let H_i $(i=1,2\ldots,n)$ be the space on which T_i acts, for each i. We note that each M_i reduces every operator $B\in W^*(T)$. Consider the mapping $\rho: B\to B\,|\,H_1$, for each $B\in W^*(T)$. Clearly ρ is a $*$-homomorphism on $W^*(T)$ into $\mathcal{B}(H_1)$. But there is a unitary operator $U_1: H_1\to M$ such that $U_1T_1U_1^*=A$. Define $\nu(D)=U_1DU_1^*$, for all $D\in\mathcal{B}(H_1)$. Then $\Pi=\nu\rho$ is a $*$-algebra homomorphism on $W^*(T)$ into $\mathcal{B}(M)$ such that $\Pi(T)=A$ and $\Pi(I)=I$. Clearly Π is an irreducible C^*-representation of $W^*(T)$ into $\mathcal{B}(M)$. It remains to show that $W^*(T)\cap\mathcal{K}(H)\not\subset\ker(\Pi)$.

Assume contrarily that $W^*(T)\cap\mathcal{K}(H)\subset\ker(\Pi)$.

(1) Then $(W^*(T)\cap\mathcal{K}(H))\,|\,H_1=\{0\}$. It is easy to verify that there is a unitary operator U_{1j} $(j=1,2,\ldots,n)$ such that $U_{1j}(B\,|\,H_1)\,U_{1j}^*=B\,|\,H_1$, for each $j=1,2,\ldots,n$, for all $B\in W^*(T)$. Hence $B\,|\,H_j=0$, for all $B\in W^*(T)\cap\mathcal{K}(H)$. It follows that

(2) $B\,|\,N=0$, for all $B\in W^*(T)\cap\mathcal{K}(H)$, where $N=\mathrm{Eig}(A\,;\,T)=H_1\oplus H_2\oplus\cdots\oplus H_n$. It is not very hard to see that $W^*(T)\,|\,H_1=\mathcal{B}(H_1)$, that is $\Pi(W^*(T))=\mathcal{B}(H_1)$. Since $\mathcal{K}(H_1)\subset\mathcal{B}(H_1)$, we can find a nonzero compact operator C_1 acting on H_1 such that $B\,|\,H_1=C_1$, for some $B\in W^*(T)$. Put $B\,|\,H_i=C_i$ $(i=1,2,\ldots,n)$. Then $C_1\simeq C_2\simeq\ldots\simeq C_n(\neq 0)$. Then $D=C_1\oplus C_2\oplus\cdots\oplus C_n\neq 0$ and $D\in\mathcal{K}(N)$. Let $P\in\mathcal{B}(H)$ be the (orthogonal) projection whose range is N. Then BP is a nonzero compact operator with $BP\in W^*(T)\cap\mathcal{K}(H)$. Now, $(BP)\,|\,N=(BP)\,|\,(\sum_{i=1}^{n}\oplus H_i)=\sum_{i=1}^{n}\oplus C_i=D\neq 0$. But $(BP)\,|\,N=B\,|\,N=0$ by (2). This is a contradiction.

(ii)$\Rightarrow$(i) and (iii). Let Π be an irreducible C^*-representation of $W^*(T)$ into $\mathcal{B}(M)$ such that $\Pi(T)=A$, $\Pi(I)=I$ and $W^*(T)\cap\mathcal{K}(H)\not\subset\ker(\Pi)$. Apply Proposition 2.5 (i) ([5], p. 333) to $W^*(T)$. Thus there exists a reducing subspace H_1 of T in H such that $U_1(B\,|\,H_1)\,U_1^*=\Pi(B)$, for all $B\in W^*(T)$, where $U_1: H_1\to M$ is a fixed unitary operator. Put $T_1=T\,|\,H_1$. Then $U_1T_1U_1^*=U_1(T\,|\,H_1)\,U_1^*=\Pi(T)=A$. Here we note that A is an irreducible operator, so we have $A\in\Sigma_0(T)$. By Proposition 2.3 ([5], p. 332), we can write $T=(T_1\oplus T_2\oplus\ldots)\oplus S$, where $T_1\simeq T_2\simeq\ldots\simeq A$ with cardinalily of $\{T_1, T_2, \ldots\}=\mathrm{mult}(A\,;\,T)$, while each $T_i \,\mathring{\delta}\, S$. We first claim that mult $(A\,;\,T)<\infty$, which proves (ii)$\Rightarrow$(i). Suppose contrarily that $\mathrm{mult}(A\,;\,T)=\infty$. Then we can find a unitary operator U such that

(3) $U\rho(B)U^*=B$, for all $B\in W^*(\infty A)$, where ρ is a *-isomorphism from $W^*(\infty A)$ onto $W^*(T_1\oplus T_2\oplus\cdots)$ such that $\rho(\infty A)=T_1\oplus T_2\oplus\ldots$and $\rho(I)=I$. To see this, let $U_i: H_i\to M$ be a unitary operator such that $U_iT_iU_i^*=A$, for all $i=1, 2, 3, \ldots$ Put $U=U_1\oplus U_2\oplus\cdots\oplus U_n\oplus\cdots : H_1\oplus H_2\oplus\cdots\oplus H_n\oplus\ldots\to M\oplus M\oplus\cdots$. Then evidently $U(T_1\oplus T_2\oplus\cdots)U^*=\infty A$ and U satisfies (3). We shall see that

(4) Any element $B\in W^*(\infty A)$ is of the form $B=\infty C$, with $C\in W^*(A)$. In fact if $B\in W^*(\infty A)$, then B is the limit of a norm bounded net $\{p_\lambda(\infty A, \infty A^*)\}_\lambda$ of polynomials with respect to the weak operator topology. But note that $p_\lambda(\infty A, \infty A^*)=\infty p_\lambda(A, A^*)$ and $\|p_\lambda(\infty A, \infty A^*)\|=\|p_\lambda(A, A^*)\|$, for each λ. Since each $\{0\}\oplus\cdots\oplus\{0\}\oplus M\oplus\{0\}\oplus\ldots$ is a reducing subspace of $p_\lambda(\infty A, \infty A^*)$, for each λ, it is also a reducing subspace of B. Thus we can decompose $B=B_1\oplus B_2\oplus\ldots$, with respect to $M\oplus M\oplus\ldots$. From this, it is immediate to see that $p_\lambda(A, A^*)\to B_i$ in the weak operator topology for M, for each $i=1, 2, \ldots$. Thus $B_1=B_i$, for all $i\geq 1$. Put $C=B_1\in W^*(A)$, then$B=\infty C$, which proves (4). But any element B, of the form $B=\infty C$, is compact if and only if $C=0$. Thus $W^*(\infty A)$ contains no nonzero compact operator. It follows that $W^*(T_1\oplus T_2\oplus\cdots)$ does not contain any nonzero compact operator. On the while,

$$W^*(T)\cap\mathscr{K}(H)\not\subset\ker(\Pi).$$

We find a compact operator $K\in W^*(T)\cap\mathscr{K}(H)$ such that $\Pi(K)\neq 0$ ...(5)
But K is also a weak operator topology limit of polynomials $p(T, T^*)$ of T, T and I, and $T=(T_1\oplus T_2\oplus\cdots)\oplus S$. It follows that $K=Q\oplus R$, for some $Q\in W^*(T_1\oplus T_2\oplus\cdots)$ and $R\in W^*(S)$. Q is also a compact operator. By what we have seen just above, we see that $Q=0$, as an operator on $H_1\oplus H_2\oplus\cdots$. But $U_1(K|H_1)U_1^*=\Pi(K)$ as we saw in the beginning of the proof for (ii)$\Rightarrow$(i) and (iii). Then $0=U_1(Q|H_1)U_1^*=U_1(K|H_1)U_1^*=\Pi(K)$, which contradicts to (5). This proves (ii)$\Rightarrow$(i). We saw (ii) implies that $A\simeq T_1$ and $\mathrm{mult}(A\,;\,T)<\infty$. By Corollary 2.7 ([5], p. 334) we see that H_1 is contained in the effective subspace of $W^*(T)\cap(H)$. This proves that (ii)$\Rightarrow$(iii).

(iii)$\Rightarrow$(ii) Let $A\in\mathscr{B}(M)$ be an irreducible operator. By restriction and then by unitary equivalence, it is clear that there is an irreducible C^*-representation Π of $W^*(T)$ into $\mathscr{B}(M)$ such that $\Pi(T)=A$, $\Pi(I)=I$. Suppose, in this case, that $W^*(T)\cap\mathscr{K}(H)\subset\ker(\Pi)$. Then $(W^*(T)\cap\mathscr{K}(H))|H_1=\{0\}$. But this cannot happen since H_1 is a nonzero subspace of the effective subspace of $W^*(T)\cap\mathcal{K}(H)$. Hence $W^*(T)\cap\mathcal{K}(H)\not\subset\ker(\Pi)$, proving (iii)$\Rightarrow$(ii). Q. E. D.

COROLLARY 1. *Let M and H be separable Hilbert spaces, with H infinite dimensional. Let $A\in\mathcal{B}(M)$ and $T\in\mathcal{B}(H)$. Suppose that there is an irreducible $*$-homomorphism $\pi: C^*(T)$ into $\mathcal{B}(M)$ such that $\pi(T)=A$, $\pi(I)=I$ and $C^*(T)\cap\mathcal{K}(H)\not\subset\ker(\pi)$. Then π is extended to an irreducible $*$-homomorphism $\Pi: W^*(T)$ into $\mathcal{B}(M)$ such that $W^*(T)\cap\mathcal{K}(H)\not\subset\ker(\Pi)$.*

Proof. By the hypotbesis and Theorem 2. 8 ([5], p. 334), we see that $A\in\Sigma_{00}(T)=\Sigma_0(T)\sim\Sigma_{ess}(T)$. It follows that mult$(A\,;\,T)<\infty$. By (i) and (ii) if Theorem 3, we see that there is an irreducible $*$-homomorphism Π of $W^*(T)$ into $\mathcal{B}(M)$ such that $\Pi(T)=A$, $\Pi(I)=I$ and $W^*(T)\cap\mathcal{K}(H)\not\subset\ker(\Pi)$. The fact that Π and π coincide on $C^*(T)$ is clear, since $\Pi(T)=\pi(T)$. Q. E. D.

COROLLARY 2. *Let M and H be separable Hilbert spaces, with H infinite dimensional. Let $A\in\mathcal{B}(M)$ and $T\in\mathcal{B}(H)$. Let $A\in\Sigma_0(T)$. Let N be a reducing subspace of T such that $A\simeq T|N$. Let Π be the $*$-homomorphism of $W^*(T)$ into $W^*(M)$ such that $\Pi(T)=A$ and $\Pi(I)=I$, by composing the restriction of $W^*(T)$ to N and the unitary equivalence from $W^*(T)|N$ onto $W^*(A)$. Then the following three conditions are equivalent.*

(i) mult$(A\,;\,T)=\infty$

(ii) $W^*(T)\cap\mathcal{K}(H)\subset\ker(\Pi)$

(iii) $(W^*(T)\cap\mathcal{K}(H))\,|\,\mathrm{Eig}(A\,;\,T)=\{0\}$.

Proof. The equivalence (i)$\Leftrightarrow$(iii) has already been seen in the proof of Theorem 3, and hence omitted.

(i)$\Rightarrow$(ii). Assume that $W^*(T)\cap\mathcal{K}(H)\not\subset\ker(\Pi)$. Then by (ii)$\Rightarrow$(i) of Theorem 3, we see that mult$(A\,;\,T)<\infty$. This is a contradiction.

(ii)$\Rightarrow$(i). Assume that mult$(A\,;\,T)<\infty$. Then we take look at proof of (i)$\Rightarrow$(ii) of Theorem 3, to get the contradiction. Q. E. D.

If we immitate the proof (i)$\Rightarrow$(iii) in Theorem 3, then the next proposition can be proven. We omit here the actual proof.

PROPOSITION 1. *Let $A\in\mathcal{B}(M)$ and $T\in\mathcal{B}(H)$, where H is an infinite dimensional separablè space. Then the following conditions are equivalent*

(i) $A\in\Sigma_{00}(T)$.

(ii) *A is an irreducible operator and it is unitarily equivalent to a sub-operator $T_1\in\mathcal{B}(H_1)$ of T, where H_1 is contained in the effective subspace of $C^*(T)\cap\mathcal{K}(H)$ (cf. [5], p. 334, Theorem 2. 8).*

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