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LOCAL CONSTANCY PRINCIPLE AND ITS APPLICATION TO THE PARTIAL DIFFERENTIAL EQUATIONS

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Introduction

In 1957, H. Lewy in his famous article on the partial differential equations without solution raised a question whether every homogeneous partial differential equation has a nonconstant solution. This question was negatively answered by L. Nirenberg in 1973 (*cf.* [3]). He showed that perturbed Mizohata operator

$$\frac{\partial}{\partial t}+it(1+\rho)\frac{\partial}{\partial x}$$

has no nonconstant solution for certain functions ρ.

This perculiar phenomena was for long time unexplained until finally in 1979, F. Treves introduced *local constancy principle* and explained the reason why Nirenberg's example has no nonconstant solutions (*cf.* [7]).

F. Treves, however, dealt with only Nirenberg's example in a generalized form; namely,

$$\frac{\partial}{\partial t}+it\{1+\rho+th(x,t)\}\frac{\partial}{\partial x}.$$

It will be shown here that if $L=\frac{\partial}{\partial t}+ib(x,t)\frac{\partial}{\partial x}$ and if $b(x,t)$ satisfies certain condition, called (M_k), with an odd integer k (*cf.* § 1), then any nonconstant solution of $Lu=0$ satisfies the local constancy principle, L can be transformed by the diffeomorphism into

$$L=\frac{\partial}{\partial t}+it^k\{1+th(x,t)\}\frac{\partial}{\partial x},$$

and, moreover, for this operator L the parallel results obtained by Treves can be generalized; that is,

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$$Lu=\frac{\partial u}{\partial t}+it^k\{1+th(x,t)\}\frac{\partial u}{\partial x}=f$$

is not locally solvable for generic f,

$$L_1u=\frac{\partial u}{\partial t}+it^k\{1+\rho+th(x,t)\}\frac{\partial u}{\partial x}=0$$

has no nonconstant solution for certain functions ρ, and with the same ρ,

$$Lu=\frac{\partial u}{\partial t}+it^k\{1+th(x,t)\}\frac{\partial u}{\partial x}=\rho u$$

has no nontrivial solution.

The class of operator satisfying (M_k) with odd integers $k\geqq 0$ includes the generalized Mizohata operator

$$\frac{\partial}{\partial t}+it^k\frac{\partial}{\partial x}$$

and $L=\frac{\partial}{\partial t}+ib(x,t)\frac{\partial}{\partial x}$ with $b(x,t)$, odd function with respect to t variable.

§1. Reduction to the canonical form

Throughout this paper Ω will stand for an open subset of R^2. We shall denote a point in R^2 by (x,t). Let L be a C^∞ complex vector field in Ω defined by

$$L=\frac{\partial}{\partial t}+ib(x,t)\frac{\partial}{\partial x}$$

where $b(x,t)$ is a real valued C^∞ function in Ω. For an integer $k\geqq 0$, we define Σ_k, a subset of Ω such as

$$\Sigma_k=\{(x,t)\in\Omega\,|\,\frac{\partial^k b}{\partial t^k}(x,t)=0\}.$$

For each integer $k\geqq 0$, we introduce a condition (M_k) on $b(x,t)$ such that

(M_k) (1) Σ_j $(j=0,1,2,\ldots,k-1)$ *are all equal and coincide with an one dimensonal* C^∞ *submanifold* Σ *of* Ω,

(2) $\frac{\partial^k b}{\partial t^k}(x,t)\neq 0$ *for any* $(x,t)\in\Sigma$.

THEOREM 1.1. *Let* $b(x,t)$ *be a real valued* C^∞ *function in* Ω. *Suppose that the property* (M_k) *holds in* Ω *for some integer* $k\geqq 0$. *If* $w_0\in\Sigma$, *then there exists an open neighborhood* U *of* w_0 *such that*

$$b(x, t) = (t-\phi(x))^k g(x, t)$$

where $\phi(x)$ is a real valued C^∞ function in Ω and $g(x, t)$ is nowhere vanishing real valued C^∞ function in U.

Proof. By the suitable translation of coordinates, we may assume that w_0 is the origin of R^2.

Since $\frac{\partial^j b}{\partial t^j}(0, 0) = 0$ for all $j=0, 1, 2, \ldots, k-1$ and $\frac{\partial^k b}{\partial t^k}(0, 0) \neq 0$, by the Malgrange preparation theorem, there exists U, an open neighborhood of w_0, such that in U

$$b(x, t) = (t^k + \alpha_1(x) t^{k-1} + \cdots + \alpha_k(x)) g(x, t)$$

where $\alpha_i(x)$ $(i=1, 2, \ldots, k)$ is a real valued C^∞ function in U and $g(x, t)$ is nowhere vanishing real valued C^∞ function in U. By the condition (M_k), for each fixed x,

$$t^k + \alpha_1(x) t^{k-1} + \cdots + \alpha_k(x)$$

has k-multiple root $t=\phi(x)$. Therefore,

$$t^k + \alpha_1(x) t^{k-1} + \cdots + \alpha_k(x) = (t-\phi(x))^k.$$

Since $-k\phi(x) = \alpha_1(x)$ and $\alpha_1(x)$ is a real valued C^∞ function, $\phi(x)$ is also a real valued C^∞ function in U. Thus

$$b(x, t) = (t-\phi(x))^k g(x, t)$$

where $\phi(x)$ and $g(x, t)$ are real valued C^∞ functions and $g(x, t)$ is nowhere vanishing in U. Q. E. D.

THEOREM 1. 2. *Suppose (M_k) holds for $b(x, t)$ in Ω. Let $w_0 \in \Sigma$. Then there exists an open neighborhood U of w_0 and a local coordinates (y, s) in U, vanishing at w_0, such that, in the local chart $(U : y, s)$*

$$L = \frac{\partial}{\partial t} + ib(x, t) \frac{\partial}{\partial x}$$

takes the form

$$L = g(y, s) \left\{ \frac{\partial}{\partial s} + is^k [1 + sh(y, s)] \frac{\partial}{\partial y} \right\}$$

where $g(y, s)$ is a nowhere vanishing complex valued C^∞ function in U and $h(y, s)$ is a real valued C^∞ function in U.

Proof. Let $w_0 = (x_0, t_0)$. By the theorem 1, there exists an open neighborhood U of w_0 such that

$$b(x,t)=(t-\phi(x))^k h(x,t).$$

Let $u=t-\phi(x)$ and $z=x-x_0$. Then

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial u},\quad \frac{\partial}{\partial x}=-\phi'(z)\frac{\partial}{\partial u}+\frac{\partial}{\partial z}.$$

Therefore, setting $\tilde{h}(z,u)=h(x,t)$, we have

$$L=\frac{\partial}{\partial u}+iu^k\tilde{h}(z,u)\left(-\phi'(z)\frac{\partial}{\partial u}+\frac{\partial}{\partial z}\right)$$

$$=(1-iu^k\tilde{h}(z,u)\phi'(z))\frac{\partial}{\partial u}+iu^k\tilde{h}(z,u)\frac{\partial}{\partial z}.$$

We shrink U, if necessary, such that in U

$$1-iu^k\tilde{h}(z,u)\phi'(z)\neq 0,$$

$$1+iu^k\tilde{h}(z,u)\phi'(z)\neq 0.$$

Since $w_0=(0,0)$ in (z,u) coordinates, this shrinking is possible. Thus in U,

$$L=(1-iu^k\tilde{h}(z,u)\phi'(z))\left(\frac{\partial}{\partial u}+\frac{iu^k\tilde{h}(z,u)}{1-iu^k\tilde{h}(z,u)\phi'(z)}\frac{\partial}{\partial z}\right)$$

$$=(1-iu^k\tilde{h}(z,u)\phi'(z))\left\{\frac{\partial}{\partial s}+\frac{iu^k\tilde{h}(z,u)[1+iu^k\tilde{h}(z,u)\phi'(z)]}{1+u^{2k}\tilde{h}^2(z,u)[\phi'(z)]^2}\frac{\partial}{\partial z}\right\}$$

$$=(1-iu^k\tilde{h}(z,u)\phi'(z))\left\{\frac{\partial}{\partial u}-\frac{u^{2k}\tilde{h}^2(z,u)[\phi'(z)}{1+u^{2k}\tilde{h}^2(z,u)[\phi'(z)]^2}\frac{\partial}{\partial z}\right.$$

$$\left.+i\frac{u^k\tilde{h}(z,u)}{1+u^{2k}\tilde{h}^2(z,u)[\phi'(z)]^2}\frac{\partial}{\partial z}\right\}$$

We perform a second change of variables $z=z(\xi,\tau)$, $u=\tau$ in U such that

$$\frac{\partial}{\partial\tau}=\frac{\partial}{\partial u}-\frac{u^{2k}\tilde{h}^2(z,u)\phi'(z)}{1+u^{2k}\tilde{h}^2(z,u)[\phi'(z)]^2}\frac{\partial}{\partial z},$$

$$\frac{\partial}{\partial\xi}=\frac{\partial}{\partial z},$$

whence

$$L=\tilde{f}(\xi,\tau)\left\{\frac{\partial}{\partial\tau}+i\tau^k\beta(\xi,\tau)\frac{\partial}{\partial\xi}\right\}$$

where

$$f(z,u)=1-iu^k\tilde{h}(z,u)\phi'(z).$$

$$\tilde{f}(\xi,\tau)=f(z(\xi,\tau),\tau), \text{ and}$$

$$\beta(\xi,\tau)=\frac{\tilde{h}(z,u)}{1+u^{2k}\tilde{h}^2(z,u)[\phi'(z)]^2}$$

Since $\beta(\xi,\tau)$ is a real valued C^∞ function in U and $\beta(w_0)\neq 0$, Shrinking U, if necessary, we have

$$\beta(\xi,\tau)=\beta(\xi,0)[1+\tau\sigma(\xi,\tau)].$$

Our last change of variables will be of the form $\xi=\xi(\zeta)$, $\tau=\sigma$, so that

$$\frac{\partial}{\partial\zeta}=\beta(\xi,0)\frac{\partial}{\partial\zeta}, \quad \frac{\partial}{\partial\tau}=\frac{\partial}{\partial\sigma}.$$

Finally we revert to the notation y, s in place of ζ, σ respectively and get

$$L=g(y,s)\left\{\frac{\partial}{\partial s}+is^k[1+sh(y,s)]\frac{\partial}{\partial y}\right\}$$

where $g(y,s)=\tilde{f}(\xi,\tau)$, $h(y,s)=\gamma(\xi,\tau)$.

In each coordinates change used in the above we may take the coordinates to vanish at w_0 by the suitable translation. Thus the coordinates (y,s) vanish at w_0. Q. E. D.

REMARK. In the new coordinates (y,s), we have

$$\Sigma\cap U=\{(y,s)\in U\,|\,s=0\}.$$

This is clear from the way of constructing new coordinates in the proof of the theorem 1. 2.

Thus by the theorem 1. 2, when (M_k) holds for $b(x,t)$, L reduces locally in the neighborhood U of $w_0\in\Sigma$ into a canonical form

$$L=g(x,t)\left\{\frac{\partial}{\partial t}+it^k[1+th(x,t)]\frac{\partial}{\partial x}\right\}.$$

We note that in this new coordinates

$$w_0=(0,0), \quad \Sigma\cap U=\{(x,t)\,|\,t=0\}.$$

§ 2. Local constancy of the solution

THEOREM 2. 1. *Suppose that (M_k) holds for $b(x,t)$ for some odd integer $k=2m+1$. If $Lu=\partial u/\partial t+ib(x,t)\partial u/\partial x=0$ has a solution $u(x,t)=A(x,t)+iB(x,t)$, where A, B are real valued C' function, in the open neighborhood V of $w_0\in\Sigma$ such that if $du(w)\neq 0$ for any $w\in V$, then for each fixed x, near to x_0, the function $B(x,\cdot)$ or $A(x,\cdot)$ attains local minimum or local maximum at $t(x)$ where $(x,t(x))\in\Sigma\cap V$.*

Proof. Let $H_u=\frac{\partial u}{\partial t}\frac{\partial}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial}{\partial t}$ be the Hamiltonian field on V associated with u. Then

$$H_u u=\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}=0.$$

Since $du=\left(\frac{\partial u}{\partial t},\ \frac{\partial u}{\partial x}\right)\neq 0$ on V and since $Lu=\frac{\partial u}{\partial t}+ib(x,t)\frac{\partial u}{\partial x}=0$, it follows that $L=g(x,t)H_u$, that is,

$$\frac{\partial}{\partial t}+ib(x,t)\frac{\partial}{\partial x}=g(x,t)\left(\frac{\partial u}{\partial t}\frac{\partial}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial}{\partial x}\right)$$

where $g(x,t)$ vanishes nowhere on V. From whence, we get

$$g(x,t)\frac{\partial u}{\partial t}=ib(x,t),\quad g(x,t)\frac{\partial u}{\partial x}=-1,$$

or,

$$g(x,t)(A_t+iB_t)=ib(x,t),\quad g(x,t)(A_x+iB_x)=-1.$$

Since $A(x,t), B(x,t)$ and $b(x,t)$ are all real valued, from the first equation we have $B_t=b(x,t)\mathrm{Re}\left(\frac{1}{g}\right)$. Since $\frac{\partial^j b}{\partial t^j}=0$ for $j=0,1,2,\ldots,k-1$ on Σ, we have $\frac{\partial^j B}{\partial t^j}=0$ for $j=1,2,\ldots,k$ on $\Sigma\cap V$. Similarly, $A_t=b(x,t)\ \mathrm{Im}\left(\frac{1}{g}\right)$. Hence $\frac{\partial^j A}{\partial t^j}=0$ for $j=1,2,\ldots,k$ on $\Sigma\cap V$.

Suppose that both $\frac{\partial^{k+1}A}{\partial t^{k+1}}(w)=0$ and $\frac{\partial^{k+1}B}{\partial t^{k+1}}(w)=0$. Then $\frac{\partial^k b}{\partial t^k}(w)=0$. Thus either $\frac{\partial^{k+1}A}{\partial t^{k+1}}(w)\neq 0$ or $\frac{\partial^{k+1}B}{\partial t^{k+1}}(w)\neq 0$ at each point w in $\Sigma\cap V$.

Since $k=2m+1$ is an odd integer, this means that for each x, near to x_0, the function $B(x,\cdot)$ or $A(x,\cdot)$ attians local minimum or local maximum at $t(x)$ where $t(x)$ is such that $(x,\ t(x))\in\Sigma\cap V$. Q. E. D

REMARK. By shrinking V, the open neighborhood of w_0, if necessary, we get either $\frac{\partial^{k+1}A}{\partial t^{k+1}}(w)\neq 0$ for any $w\in\Sigma\cap V$ or $\frac{\partial^{k+1}B}{\partial t^{k+1}}(w)\neq 0$ for any $w\in\Sigma\cap V$, since $\frac{\partial^{k+1}A}{\partial t^{k-1}}$ or $\frac{\partial^{k+1}B}{\partial t^{k+1}}$ is a continuous function. Moreover, since $\Sigma=\{(x,t)\in\Omega\,|\,b(x,t)=0\}$, and snce $b(x,t)=(t-\phi(x))h(x,t)$ where $h(x,t)$ nowhere vanishes, $b(x,t)$ has the same sign in the one side of Σ in U. Thus shrinking V, if necessary, either $A(x,\cdot)$ or $B(x,\cdot)$ attains global

minimum along $\Sigma \cap V$ in V or global maximum along $\Sigma \cap V$ in V. Therefore, taking $iu(x, t)$, which is also a solution of $Lu=0$ if $u(x, t)$ is, or changing sign of $u(x, t)$ or $iu(x, t)$, we may assume that $B(x, \cdot)$ attains minimum along $\Sigma \cap V$.

Thus, in the sequel, *we shall assume that* $B(x, \cdot)$ *attains minimum along* $\Sigma \cap V$ *in* V.

DEFINITION Let D be a subset of R^2 and C be the field of complex numbers. Let u be a mapping from D into C. The *fiber* of u at $z \in u(D)$ is the set of points (x, t) in D such that $u(x, t)=z$.

THEOREM 2.2. *Let* $b(x, t)$ *satisfy* (M_k) *for an odd integer* $k \geqq 0$. *Suppose that* $u(x, t)=A(x, t)+iB(x, t)$ *is the solution of*

$$\frac{\partial u}{\partial t}+ib(x, t)\frac{\partial u}{\partial x}=0$$

such that $du \neq 0$ *in the open neighborhood* V *of* $w_0 \in \Sigma$. *We assume that* $B(x, \cdot)$ *attains minimum along* $\Sigma \cap V$.

Then one can choose a new local coordinates (y, s) *in a simply connected open neighborhood* $U \subset V$ *of* w_0, *vanishing at* w_0, *such that the following local constancy principle holds;*

(*L. C.*) *the fibers of* u *in* U *consists either of a single points, when they are contained in* $U \cap \Sigma$, *or else of a pair of points,* (y, s) *and* $(y, -s)$, *when they do not intersect* Σ.

Proof. In this proof, we take the advantage of the reduced form of L as is done in the theorem 1.2. Thus we assume that $b(x, t)=t^k(1+th(x, t))$ and u is the solution of

$$\frac{\partial u}{\partial t}+it^k[1+th(x, t)]\frac{\partial u}{\partial x}=0.$$

We shall first show that A_x does not vanish in V. In fact, $Lu=0$ implies that $(A_t+iB_t)+ib(x, t)(A_x+iB_x)=0$ or, $B_t=b(x, t)A_x$. Note that $\frac{\partial^j B}{\partial t^j}(x, t)=0$ for $j=1, 2, \ldots, k$ and $\frac{\partial^j b}{\partial t^j}(x,t)=0$ for $j=0,1,2, \ldots, k-1$ only in $\Sigma \cap V$ and $\frac{\partial^{k+1} B}{\partial t^{k+1}} \neq 0$ and $\frac{\partial^k b}{\partial t^k} \neq 0$ in $\Sigma \cap V$. But $A_x(w)=0$ for $w \in V/\Sigma$ implies $B_t(w)=0$ for $w \in V/\Sigma$, while $A_x(w)=0$ in $\Sigma \cap V$ implies $\frac{\partial^{k+1} B}{\partial t^{k+1}}(w)=0$ for $w \in \Sigma \cap V$ Both cases cannot occur. Thus $A_x(w) \neq 0$ for any $w \in V$.

In regard of the reduced form of L, that $A_x(w) \neq 0$ for each $w \in \Sigma \cap V$

implies that the set $K_\xi = \{(x,t) \mid A(x,t)=\xi\}$ for a real number ξ is one dimensional manifold which is transversal to Σ, whenever K_ξ meets with Σ, in the neighborhood of $\Sigma \cap V$. We may take V sufficiently small and thus may assume K_ξ is one dimensional submanifold of V which is transversal to Σ whenever K_ξ meets with Σ.

Now let us notice that, by shrinking V, if necessary, we may assume that $b(x,t)<0$ for (x,t) in $V^+ = \{(x,t) \in V \mid t>0\}$ and $b(x,t)>0$ for (x,t) in $V^- = \{(x,t) \in V \mid t<0\}$

Let $x=x(t)$ be the solution of $A(x,t)=\xi$ for some real number ξ. Then $x=x(t)$ is the solution of

$$A_x \frac{dx}{dt} + A_t = 0 \quad \text{or} \quad x'(t) = -\frac{A_t}{A_x},$$

since A_x does not vanish on V. Let $Y(x,t) = -\frac{A_t}{A_x}$. Then if $b(x,r)<0$ for any $(x,r) \in V$, $b(x,t) \leqq 0$ for any $(x,t) \in V$ with $t \geqq r$. Also since $A_t = 0$ on $\Sigma \cap V$ (*cf.* Proof of theorem 2.1), $Y(x,t)=0$ at any point $(x,t) \in \Sigma \cap V$ where $b(x,t) = b_t(x,t) = 0$. Therefore by the H. Brezis lemma (*cf.* [1]), if $b(x(t_0), t_0) < 0$, then $b(x(t), t) \leqq 0$ for $t \geqq t_0$. Applying the same arguments for $-b(x,t)$, we conclude that $B(x(t), t)$ attains global minimum in V at $t=0$.

Thus $A(x,t) = y + u(w_0)$, $B(y,t) = s + B(y,0)$ gives a local coordinates (y,s) in some open neighborhood U of w_0 such that local constancy principle holds. It is clear that we may take U to be simply connected. Q. E. D.

REMARK. Let us remark that the open neighborhood U has to be taken to be *u-symmetry* in the sense that $(y,s) \in U$ if and only if $(y,-s) \in U$.

REMARK. Consider $u(x,t) = A(x,t) + iB(x,t)$ as a map from V to C. We notice that the Jacobian determinant of $A = \mathrm{Re}(u)$ and $\mathrm{Im}(u)$ with respect to x, t is as same as $\{A, B\} = \frac{1}{2i}\{u, \bar{u}\}$ where $\{P, Q\}$ indicates the *Poisson bracket*; i. e.,

$$\{P, Q\} = \frac{\partial P}{\partial X}\frac{\partial Q}{\partial t} - \frac{\partial P}{\partial t}\frac{\partial Q}{\partial x}.$$

Now since u and $\bar{u}$ are nonconstant solutions of $Lu=0$ and $Lu=0$, as in the proof of the theorem 2.1

$$L = g(x,t) H_u, \quad \bar{L} = g(x,t) H_{\bar{u}}.$$

It is obvious that $L = \frac{\partial}{\partial t} + ib(x,t)\frac{\partial}{\partial x}$ and $\bar{L} = \frac{\partial}{\partial t} - ib(x,t)\frac{\partial}{\partial x}$ are C-line-

arly independent in $V\setminus\Sigma$ and C-linearly dependent in $\Sigma\cap V$. Thus the pair gH_u and $\bar{g}H_u$ are C-linearly independent in $V\setminus\Sigma$ and C-linearly dependent in $V\cap\Sigma$.

Since $gH_{\bar{u}}=g\left(\frac{\partial u}{\partial t}\frac{\partial}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial}{\partial t}\right)$, $\bar{g}H_{\bar{u}}=g\left(\frac{\partial \bar{u}}{\partial t}\frac{\partial}{\partial x}-\frac{\partial \bar{u}}{\partial x}\frac{\partial}{\partial t}\right)$ and $g(x,t)$ does not vanish in V, it follows that $\frac{1}{2i}\{u,\bar{u}\}=\{A,B\}$ vanishes on $\Sigma\cap V$ but never vanishes on $V\setminus\Sigma$.

Therefore, for any real ξ,ζ, the set $K\xi=\{(x,t)\mid A(x,t)=\xi\}$ is transversal to the set $\tilde{K}_\zeta=\{(x,t)\mid B(x,t)=\zeta\}$ whenever they meet in $V\setminus\Sigma$.

§ 3. Preliminary theorems.

The following three theorems are due to F. Trever (*cf.* [7]). We offer his proofs to make this article self contained and to use his results in later sections.

THEOREM 3. 1. *Let $\{U,\Sigma\}$ satisfy local constancy principle with respect to a complex valued function $u(x,t)$. Let V be an open subset of U, u-symmetric and connected. Let $h\in C'(V)$ be such that $d(hdu)=0$. Then h is constant on each fiber of u in V.*

Proof. Let $U^+=\{(x,t)\in U\mid t>0\}$ and $U^-=\{x,t)\in U\mid t<0\}$. Set $V^\pm=V\cap U^\pm$, $V_0=V\cap\Sigma$, $Q=\mu(V^+)=u(V^-)$, $\Gamma=\mu(V_0)$. Call h^+ the restriction of h to V^+, h^- its restriction to V^-. Since u is a diffeomorphism of V^+ or V^- onto Q, we may form $\tilde{h}^\pm=\tilde{h}^\pm\cdot u^{-1}$, and $\tilde{h}=\tilde{h}^+-\tilde{h}^-$, which are functions in Q. Since $d(hdu)=0$ in V we have $d(\tilde{h}(z)dz)=0$ in $Q:\tilde{h}$ is holomorphic in Q. But $\tilde{h}(z)\to 0$ as $z\to\Gamma$, and Γ is a smooth curve, containing nonempty open arcs, and part of the foundary of Q. This implies that $\tilde{h}\equiv 0$ in Q. Q. E. D.

THEOREM 3. 2. *Let U, V be as above. Suppose furthermore that V is simply connected. Let $g\in C^\infty(V)$ be such that $d(gdu)\equiv 0$ in the complement of a compact subset K of U^+. Then*

$$\int_V d(gdu)=0.$$

Proof. We may assume that $W^+=V^+\setminus K$ is connected. Let W^- be the subset of V^- such that $W^+\cap W^-$ is u-symmetric, and set $W=W^+\cup W^-\cup V_0$: W is open, connected, u-symmetric and $d(gdu)=0$ in W, therefore g is constant on the fibers of u in W. Let γ^+ be a smooth curve in W^+, winding around K once and denote by γ^- its reflection across Σ. Thus γ and γ^- is u-symmetric. With the appropriate orientation we have

$$\int_{\gamma^+} gdu = \int_{\gamma^-} gdu.$$

But γ^- is contractible to a point in V^-, where $d(gdu)=0$, therefore $\int_{\gamma^-} gdu = 0$. Since the support of gdu is contained in the interior of γ^+, Stoke's theorem implies $\int_V d(gdu)=0$. Q. E. D.

THEOREM 3. 3. *Let L be a C^∞complex vector field in an open subset U in R^2 such that L does not vanish at any point in Ω. Let ${}^tL$ be the transpose of L and $c=L+{}^tL$. Then there is an open neighborhood U_0 of a point w_0 in Ω and a C^∞ function u in U_0 such that $w\in U_0$, $du(w)\neq 0$ and $Lu(w)=0$ if and only if there is an open neighborhood V_0 of w_0 in Ω and a function $v\in C^\infty(V_0)$ satisfying $Lv=c$ in V_0.*

Proof. The condition is necessary, for if $Lu=0$ has nonconstant solution, then $L=gH_u$ in U_0, with $g\in C^\infty(U_0)$, nowhere zero, and H_u, the Hamiltonian field of u. But then ${}^tL=-L=-(H_ug)$, hence$c=-H_ug$. Take V_0 small enough so that log g is defined in V_0 and belongs to $C^\infty(U_0)$. Then

$$L[-\log g]=-\frac{1}{g}Lg=-H_ug=c.$$

Conversely, let v be as in the statement and set $M=e^vL$. We have $Mv=L(e^v)=ce^v$, and

$${}^tM=+Le^v=-Le^v+ce^v=-M.$$

If $M=A(x,t)\frac{\partial}{\partial x}-B(x,t)\frac{\partial}{\partial t}$, we have $\frac{\partial A}{\partial x}=\frac{\partial B}{\partial t}$ everywhere in V_0. We may suppose that V_0 is simply connected and therefore there is $u\in C^\infty(V_0)$ such that $du=Bdx+Adt$, hence $M=H_u$ and therefore $Lu=0$, $du\neq 0$ at every point of V_0. Q. E. D.

§4 Local unsolvability

Let L be a complex vector field satisfying (M_k) where k is a nonnegative odd integer. Thus we may assume that

$$L=g(x,\ t)\left\{\frac{\partial}{\partial t}+it^k(1+th(x,\ t))\frac{\partial}{\partial x}\right\}$$

on an open neighborhood U of a point $w_0\in\Sigma$. Note that $w_0=(0,0)$ in (x,t) coordinates. We call U^+ (resp. U^-) the subset of U where $t>0$ (resp. $t<0$).

We denote by $\{K_\nu\}$ $\nu=1,2,\ldots$ an infinite sequence of compact subsets of

U^-, having the following properties: The sets K_ν converge to the set $\{0\}$, and, moreover, the projections on the x-axis of the K_ν are pairwise disjoint.

Below, f will denote any C^∞ function in U having the following three properties;

(i) $f \geqq 0$ everywhere,

(ii) $f \equiv 0$ in the complement of UK_ν, and

(iii) for any ν, there exists $w_\nu \in K_\nu$ such that $f(w_\nu) > 0$.

THEOREM 4. 1. *Suppose that there exists an open neighborhood* $W \subset U$ *of* $w_0 \in \Sigma$ *and a function* $v \in C^\infty(W)$ *such that in* W

$$Lv = -it^{k+1} g h_x.$$

Then there does not exist a C^∞ *function* w *in* W *such that* $Lw = f$ *in* W.

Proof. Let us write $L_0 = g^{-1}L$. Then

$$L_0 + {}^tL_0 = -it^{k+1} h_x.$$

By hypothesis, there exists a function $v \in C^\infty(W)$ such that $L_0 v = -it^{k+1} h_x$. We may apply the theorem 3. 3. We conclude that for L_0 and hence for L, there exists U_0, an open neighborhood of w_0 and a C^∞ function u in U_0 such that $w \in U_0$, $du(w) \neq 0$ and $Lu(w) = 0$. Thus we may write in U_0, $L = gH_u$ where $H_u = \frac{\partial u}{\partial t}\frac{\partial}{x} - \frac{\partial u}{\partial x}\frac{\partial}{\partial t}$ and $g \in C^\infty(U_0)$, nowhere zero.

Suppose $Lw = f$ in W. Then $H_u w = \frac{f}{g}$ in $U_0 \cap W$. Note that $(H_u w)\,dx\,dt = d(wdu)$. We select an open neighborhood $\tilde{U}$ of w_0, $\tilde{U} \subset U_0 \cap W$ and $\tilde{U}$ u-symmetric. We write $\tilde{U}^\pm = \tilde{U} \cap U^\pm$. Clearly, for ν large enough we can find an open rectangle

$$V_\nu^+ = \{(x, t) \in \tilde{U} \mid a_\nu < x < b_\nu,\ 0 < t < T_\nu\}$$

contained in $\tilde{U}^+$, containing K_ν and not intersecting any $K_{\nu'}$ for $\nu' \neq \nu$. Let V_ν^- be the unique open subsets of $\tilde{U}^-$ such that $V_\nu^+ \cup V_\nu^-$ is u-symmetric, and call I_ν the interval in Σ, $a_\nu < x < b_\nu$, $t = 0$. Then

$$V_\nu = V_\nu^+ \cup I_\nu \cup V_\nu^-$$

has the following properties: V_ν is connected and simply connected; $K_\nu \subset V_\nu$; V_ν is u-symmetric. Let w_ν denote the restriction of w to V_ν. By our hypothesis about f, the support of $d(w_\nu du)$ is contained in K_ν. Therefore,

$$\iint_{K_\nu} \frac{f}{g}\,dx\,dt = 0.$$

But as $\nu \to +\infty$ the argument of g in K_ν becomes arbitrarily close to

$\operatorname{Arg} g(0,0)$, since $f \geqq 0$ everywhere in K_ν and $f>0$ at some point of K_ν, the above is not possible as soon as ν is large enough. Thus w satisfying $Lw=f$ in W can not exist. Q. E. D.

§ 5. Nonexistence of nonconstant solutions

Let $\{K_{m,n,p}\}$ (m, n, p are positive integers) be a triple sequence whose elements are contained in $U^+$and whose x-projections are pairwise disjoint. We make the following hypothesis:

For fixed m, n and for $p \to +\infty$, the sets $K_{m,n,p}$ converges to the set consisting of a single point $(x_{m,n}, t_{m,n})$, with $t_{m,n}>0$; for fixed m, the point $(x_{m,n}, t_{m,n})$ converges to (x_m, t_m), with $t_m>0$; $\lim_{m\to\infty}(x_m, t_m)=(0,0)$.

We define then ρ in the same manner f in § 4 but relative to the $K_{m,n,p}$:

(i) $\rho \geqq 0$ everywhere:

(ii) $\rho \equiv 0$ in the complement of $\bigcup_{m,n,p} K_{m,n,p}$;

(iii) $\forall_{m,n,p}$, $\exists w_{m,n,p} \in K_{m,n,p}$ such that $\rho(w_{m,n,p})>0$.

THEOREM 5. 1. *Suppose that*

$$\frac{\partial u}{\partial t}+it^k(1+th(x, t))\frac{\partial u}{\partial x}=0$$

has a solution $u(x,t)$ in an open neighborhood U_0 of apoint $w_0 \in \Sigma$ such that $w \in U_0$, $du(w) \neq 0$. Then there is an open neighborhood $W \subset U_0$ of w_0 such that any solution $u_1 \in C^\infty(W)$ of

$$\frac{\partial u_1}{\partial t}+it^k(1+\rho+th)\frac{\partial u_1}{\partial x}=0$$

is constant in W.

Proof. We use the same notation as in the proof of the theorem 4. 1 except that ν is replaced everywhere by (m, n, p). We duplicate the reasoning in that proof, taking now $f=-it^k g\rho\frac{\partial u_1}{\partial x}$. We reach the conclusion that for m_0, n_0, p_0 large enough and $m>m_0, n>n_0, p>p_0$,

$$\iint_{m,n,p} g^{-1}t^k g\rho\frac{\partial u_1}{\partial x}dx\ dt=0.$$

But here we cannot avail ourselves of the fact $\frac{\partial u_1}{\partial x}\neq 0$.

By letting p go to $+\infty$, we derive that

$$\frac{\partial u_1}{\partial x}(x_{m,n},\ t_{m,n})=0$$

and therefore,

$$du_1(x_{m,n},\ t_{m,n})=0.$$

Now we let n go to $+\infty$. We conclude that du_1 vanishes of infinite order at (x_m, t_m). Since L_1 is e[[iptic in the region $t>0$, we conclude that $du_1\equiv 0$ for $t>0$. To derive that $du_1\equiv 0$ throughout its domain, we use the fact that, by our construction of the $K_{m,n,p}$, there is a connected u-symmetric open subset W, V, in which $L_1=\frac{1}{g}L$ and thus $Lu_1=0$, with V intersecting both U^+ and U^-. Since u_1 is constant on the fibers of u and is constant in $V\cap U^+$, we must have $du_1\equiv 0$ in V and by the ellipticity of L for $t<0$, also in $W\cap U^-$. Q. E. D.

THEOREM 5. 2. *If* $\frac{\partial u}{\partial t}+it^k(1+th\ (x,t))\frac{\partial u}{\partial x}=0$ *has a solution* u *in an open neighborhood* U_0 *of a point* $w_0\in\Sigma$ *such that* $w\in U_0$, $du(w)\neq 0$, *then the equation*

$$\frac{\partial u}{\partial t}+it^k(1+th(x,\ t))\frac{\partial u}{\partial x}+\rho u=0$$

has no nontrivial solution in some open neighborhood $W\subset U_0$ *of* w_0.

Proof. Duplicate the proof of the theorem 5. 1 with $f=\rho u_1$. Q. E. D.

REMARK. When $h(x,t)$ is analytic, then by the Cauchy-Kovalewska theorem

$$\frac{\partial u}{\partial t}+it^k(1+th(x,\ t))\frac{\partial u}{\partial x}=0$$

has always nonconstant solution required as in the theorems 5. 1 and 5. 2.

NOTICE. The second author initiated the materials in this article. More extensive and generalized version of this article will be appear in his paper now under preparation. The proofs in sections 4 and 5 are essentially duplication of Treves' proof when $k=1$.

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