韓國數學教育學會誌〈數學教育〉
1980.4. Vol. XVIII No.1

Journal of the Korea Society of Mathematical Education
April. 1980. Vol. XVIII. No.1

LIBERAL EXTENSION OF RINGS

By Lee Chong Yun

Kyung-Nam University, Masan, Korea.

1. Introduction

Let R, S be rings with identity such that $R \subseteq S$.

We call S a liberal extension of R if there is a finite set of elements $\{a_1, a_2, \cdots, a_n\}$ of S such that $S = \sum a_i R$ and $a_i r = r a_i$ for each i and for all $r \in R$.

In this section we consider the relationship between the two rings R, T such that $R \subseteq T \subseteq M_n(R)$ where $M_n(R)$ is the $n \times n$ matrix ring over R.

S is a normalizing extension of R such that $S = \sum a_i R$ and $a_i R = R a_i$ for $1 \leqslant i \leqslant n$. There exists ring T such that $R \subseteq T \subseteq S$ and $T \subseteq S = M_n(R)$ for $n \in Z$.

The result is frequently used in this paper.

2. Intermediate matrix rings

Before outlining the results it seems worthwhile to indicate some properties of liberal extensions which are commonly encountered.

$R[G]$ has krull dimension if and only if R has krull dimension. by I.G. Connel.

S has krull dimension if and only if R has krull dimension by B. Lemonnier.

S is right Noetherian if and only if R is right Noetherian by E. Formanek- A. Jategaonkar.

If K- dim $R[G]$ exists with $|G| < \infty$, then G satisfies A.C.C. on finite subgroup by S.M. Woods.

Definition 2.1

A left module M is faithful if its left annihilator (O : M) is O.

A ring R is left primitive if there exists a simple faith left R- module.

Proposition 2.2

A prime ring with a faithful module of finite length is primitive.

Proof

Suppose M_R has a faithful with finite length. There must be $M=M_0\supset M_1\supset M_2\supset\cdots\supset M_{n-1}\supset M_n=\{0\}$

Let $\{M_i\}$ be an n-series for M and set

$A_1=(0:M/M_1)$, $A_2=(0:M_1/M_2),\cdots,A_{n-1}=(0:M_{n-1}/M_n)$.

Since M is faithful, $A_1\neq 0$, $A_2\neq 0,\cdots,A_{n-1}\neq 0$ and $MA_1A_2\cdots A_{n-1}=0$ imply that $A_1A_2\cdots A_{n-1}=0$ Moreover, this implies that $A_i=0$ for $i\leqslant n-1$.

It easily follow from this fact that $A_i=(0:M_i/M_{i+1})=0$. Hence M_i/M_{i+1} is a faithful irreducible R-module. We conclude that R is primitive.

Theorem 2.3

Let R, T be ring such that $R\subseteq T\subseteq M_n(R)$ where $M_n(R)$ is the matrix ring over R.

T has krull dimension if and only if R has krull dimension. In this case

K dim $T=K$ dim R.

Proof

Necessity. If a ft $\sigma:\mathcal{L}_r(R)\rightarrow\mathcal{L}_r(T)$ is defined by $\sigma(I)=IT$, then σ is injective. Suppose $IT=JT$.

We have that $I=IT\cap R=JT\cap R=J$, thus σ is injective. Suppose that K dim T exists. We get K dim $R\leqslant K$ dim T, K dim R exists.

Sufficiency. K dim ${}_RR$ exists. We easily get that K $\dim_R M_n(R)=K$ $\dim_R R$, since$R\subseteq T\subseteq M_n(R)$ K $\dim_R$ $T\leqslant K$ $\dim_R M_n(R)=K$ $\dim_R R$, K $\dim_R$ T exists.

Reference

〔1〕. T.W. Hungerford, Algebra, Holt Rinehart and Winston, Inc, New York, 1974.

〔2〕. J.C. Robson and L.W. Small, Liberal extensions, school of mathematics, university of Leeds.

〔3〕. S.M. Woods, Existence of krull dimension in group rings, London Math. Soc, 1973.

〔4〕. J.W. Fisher, C. Lanski and J.K. Park, Gabriel dimension of finite normalizing extensions, to appear.