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Locally s-closed Spaces

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§1. Introduction

The object of this paper is to introduce a locally s-closed space. We give several characterizations of such spaces, some of which make use of filters, and a type of convergence we define as s-convergence. We Obtain the following results:

(1) Each locally s-closed, first countable, regular compact space is finite.

(2) Each extremally disconnected locally compact space is locally s-closed.

§2 Definitions and theorems.

Definition 2-1. A Hausdorff space X is H-closed if and only if for every open cover $\{U_a | a \in \Lambda\}$ there exists a finite subfamily $\{U_{a_i} | i=1, 2, \cdots, n\}$ such that the union of their closures cover X.

Definition 2-2. A set A in a topological space X is semiopen if and only if there exists an open set V such that $V \subset A \subset \bar{V}$, where $\bar{V}$ is the closure of V.

Definition 2-3. A filterbase $\mathcal{F} = \{A_a\}$ s-converges to a point $x_0 \in X$ if for each semiopen set V containing x_0 there exists an $A_a \in \mathcal{F}$ such that $A_a \subset \bar{V}$.

Definition 2-4. A filterbase $\mathcal{F} = \{A_a\}$ s-accumulates to a point $x_0 \in X$ if for each semiopen set V containing x_0 and $A_a \in \mathcal{F}$, $A_a \cap \bar{V} \neq \phi$.
The corresponding definitions using nets are apparent and will not be stated. An easy consequence of these definitions is

Theorem 2-1. Let $\mathcal{F}$ be a maximal filterbase in X. Then $\mathcal{F}$ s-accumulates to a point $x_0 \in X$ if and only if $\mathcal{F}$ s-converges to x_0. [4. Theorem 1]

Definition 2-5. A topological space X is s-closed if and only if for every semiopen cover $\{U_a | a \in \Gamma\}$ of X there exists a finite subfamily $\{U_{a_i} | i=1, 2, \cdots, n\}$ such that the union of their closures cover X.
It is apparent from the definition above that a Hausdorff s-closed space is H-closed.

Definition 2-6. A space X is locally s-closed if for each point $x \in X$ and each open cover $\{U_a | a \in I\}$ of a nighborhood $N(x)$ of x, there exists a finite subcollection $\{U_{a_i} | i=1, 2, \cdots, n\}$

such that $N(x) \subset \bigcup_{i=1}^{n} \overline{U}_{a_i}$.

Lemma. If a subset B of X is semiopen then it is semiclosed.

(Proof) Since B is semiopen, there is a open set V such that $V \subset B \subset \overline{V}$.

Since $CV = CV \supset CB \supset C\overline{V} = (CV)^{\circ} = CV$, *i.e.* $CV \supset CB \supset CV$, CB is semiopen.

Thus B is semiclosed. *Q. E. D.*

Theorem 2-2. Let X be a topological space and $N(x)$ be a neighborhood of x in X. Then (a) If $\mathcal{F}$ is a filterbase in $N(x)$ such that $\mathcal{F}$ s-converges to $y \in N(x)$, than $\mathcal{F}$ s-accumulates to y.

(b) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two filters in $N(x)$ and suppose $\mathcal{F}_2$ is stronger than $\mathcal{F}_1$. If $\mathcal{F}_2$ s-accumulates to $y \in N(x)$, then $\mathcal{F}_1$ s-accumulates to y.

(c) Let $\mathfrak{M}$ be a maximal filterbase in $N(x)$. Then $\mathfrak{M}$ s-accumulates to $y \in N(x)$ if and only if $\mathfrak{M}$ s-converges to y.

The proof of the above theorem is trivial.

Theorem 2-3. For a topological space the fellowing are equivalent:

(a) X is locally s-closed.

(b) For each point x in X and each collection of nonempty semiclosed sets $\{F_a \mid a \in I\}$ such that $(\bigcap_a F_a) \cap N(x) = \phi$ for a neighborhood $N(x)$ of x, there is a finite subcollection $\{F_{a_i} \mid i = 1, 2, \cdots, n\}$ such that $(\bigcap_{i=1}^{n} (F_{a_i})^{\circ}) \cap N(x) = \phi$.

(c) For each point x in X and each collection of nonempty semiclosed sets $\{F_a \mid a \in I\}$, if each finite subcollection $\{F_{a_i} \mid i = 1, 2, \cdots, n\}$ has the property that $(\bigcap_{i=1}^{n} (F_{a_i})^{\circ}) \cap N(x) \neq \phi$ for each neighborhood $N(x)$ of x, then $(\bigcap_a F_a) \cap N(x) \neq \phi$.

(d) For each point $x \in X$ and each filterbase $\mathcal{F} = \{N_a \mid a \in I\}$ in a neighborhood $N(x)$ of x, there exists a $y \in N(x)$ such that $\mathcal{F}$ s-accumulates to $y \in N(x)$.

(e) For each point $x \in X$ and maximal filterbase $\mathfrak{M} = \{N_a \mid a \in I\}$ in a neighborhood $N(x)$ of x there exists a $y \in N(x)$ such that $\mathfrak{M}$ s-converges to $y \in N(x)$.

(Proof) (a)$\Rightarrow$(e): Suppose that for each point $x \in X$ and maximal filterbase $\mathfrak{M} = \{N_a \mid a \in I\}$ in a neighborhood $N(x)$ of x, $\mathfrak{M}$ does not s-converge to any point in $N(x)$: therefoer, by Theorem 2-2, $\mathfrak{M}$ does not s-accumulate any point in $N(x)$.

This implies that for every $y \in N(x)$, there exists a semiopen set $V(y)$ containing y and $N_{a(x)} \in \mathfrak{M}$ such that $N_{a(x)} \cap \overline{V(y)} = \phi$. Obviously $\{V(y) \mid y \in N(x)\}$ is a semiopen cover of $N(x)$ and by hypothesis there exists a subfamily $\{V(y_i) \mid i = 1, 2, \ , n\}$ such that $\bigcup_{i=1}^{n} \overline{V(y_i)} \supset N(x)$. Since $\mathfrak{M}$ is a filterbase in $N(x)$, there exists an $N_0 \in \mathfrak{M}$ such that $N_0 \subset \bigcap_{i=1}^{n} N_{a(x_i)}$. Hence $N_0 \cap \overline{V(y_i)} = \phi$, contradicting the essential fact that $N_0 \neq \phi$.

(e)$\Rightarrow$(d): Since every filterbase in $N(x)$ contained in a maximal filterbase in $N(x)$ and since the maximal filterbase s-accumulates to some point in $N(x)$, the given filterbase s-accumulates to the point by Theorem 2-2. (b).

(d)$\Rightarrow$(b): Suppose that for each point $x \in X$ and each collection of nonempty semiclosed sets $\{F_a : a \in I\}$ such that $(\bigcap_a F_a) \cap N(x) = \phi$ for a neighborhood $N(x)$ of x, and suppose that for every finite subfamily $\{F_{a_i} | i=1, 2, \cdots, n\}$, $N(x) \cap (\bigcap_{i=1}^{n} (F_{a_i})^\circ) \neq \phi$.

Then $\mathcal{F} = \{N(x) \cap (\bigcap_{i=1}^{n} (F_{a_i})^\circ) | n \in Z^+, F_{a_i} \in \{F_a | a \in I\}\}$ forms a filterbase in $N(x)$. By hypothesis, $\mathcal{F}$ s-accumulates to some point $y_0 \in N(x)$. This implies that for every semiopen set $V(y_0)$ containing y_0, $(N(x) \cap (F_a)^\circ) \cap \overline{V(y_0)} \neq \phi$ for every $a \in I$. Since $y_0 \notin \bigcap F_a$, there exists an $a_0 \in I$ such that $y_0 \notin F_{a_0}$. Hence y_0 is contained in the semiopen set $X - F_{a_0}$. Thus $(N(x) \cap (F_{a_0})^\circ) \cap \overline{(X - F_{a_0})} = (N(x) \cap (F_{a_0})^\circ) \cap (X - (F_{a_0})^\circ) = \phi$ contradicting the fact that $\mathcal{F}$ s-converges to y_0.

(b)$\Leftrightarrow$(c): is the contraposition.

(b)$\Rightarrow$(a): Let $\{V_a | a \in I\}$ be a semiopen covering of a neighborhood $N(x)$ of x in X. Then $N(x) \cap (\bigcap_{a \in I} (X - V_a)) = \phi$. By hypothesis, there exists a finite subfamily

$\{V_{a_i} | i=1, 2, \cdots, n\}$ such that $N(x) \cap \bigcap_{i=1}^{n} (X - V_{a_i})^\circ) = N(x) \cap (\bigcap_{i=1}^{n} (X - \bar{V}_{a_i})) = \phi$.

Therefore $N(x) \subset \bigcup_{i=1}^{n} \bar{V}_{a_i}$ and consequently X is locally s-closed. Q. E. D.

Lemma 2-4. Each s-closed first countable, regular space is finite. [4, Theorem 3]

Theorem 2-5. Each locally s-closed, first countable, regular compact space X is finite.

(Proof) Since X is locally s-closed, each point x in X has an open neighborhood $N(x)$ which is s-closed, first countable, regular.
On the while, since X is compact, the cover $\{N(x) : x \in X\}$ of X has a finite subfamily $\{N(x_i) | i=1, 2, \cdots, n\}$ such that $X = \bigcup_{i=1}^{n} N(x_i)$. Since by the lemma 2-4, each $N(x_i)$ is finite, so is X. Q. E. D.

Corollary. Each locally s-closed, compact metrizable space is finite. Since every compact, regular space is metrizable, we have from the corollary of theorem 2-5. the following.

Corollary. Each finite, locally s-closed, regular compact space is uncountable.

Definition 2-7. X is called extremally disconnected if the closure of an open set is an open set.

Theorem 2-6. Each extremally disconnected, locally compact space is locally s-closed.

(Proof) If X is extremally disconnected, then the closure of an open set is an open set. The interior of a semiopen set is dense in it. We consider $\{\overline{U_t^\circ} | t \in T\}$ in stead of given semiopen cover of $N(x)$. Q. E. D.

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