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On the Maximal Ideal In $\mathrm{Hom}_{\Lambda}(M, M)$

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1. Introduction.

The present note is concerned with the maximal ideal of $\mathrm{Hom}_{\Lambda}(M, M)$, where M is an indecomposable left Λ-module with finite length. Let Λ be a ring with unit element, and let M be a left Λ-module. If it is impossible to write $M=M_1 \oplus M_2$, where M_1, M_2 are nonzero left Λ-submodules of the left Λ-module M. Then M is called indecomposable. A finite strictly descending chain of left Λ-submodules of the left Λ-module M

$$M=M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_r=0$$

is a Jordan-Hölder series in which each M_i is a maximal left Λ-submodule of M_{i-1} $i=1, 2, 3, \cdots r$.

And then r is called the finite length of M.

The other terminologies and notations are based on Szc-Tsen Hu [2].

Definition 1. A left Λ-module M is called *left Noetherian* iff every nonempty collection of left Λ-submodules of M has a maximal element.

Definition 2. A left Λ-module M is called *left Artinian* iff every nonempty collection of left Λ-submodules of M has a minimal element.

2. Proof of Theorems.

Lemma 1. *Let M be a left Λ-module and N a left Λ-submodule of M. Then M is left Artinian if and only if N and M/N are left Artinian.*

Proof. (Necessity) Any left Λ-submodule of N is a left Λ-submodule of M, so N is left Artinian.

Let $f: M \longrightarrow M/N$ be the canonical projection.

Let $M_1' \supseteq M_2' \supseteq \cdots$ be a descending sequence of left Λ-submodules of M/N.

Then we get a descending sequence of $M_1 \supseteq M_2 \supseteq \cdots$ of left Λ-submodules of M by letting $M_i = f^{-1}(Mi')$.

Hence there exists an integer k such that $M_i = M_k$ for $i \geq k$.

Therefore $M_i' = M_k'$ for all $i \geq k$ and so M/N is Noetherian. (Sufficiency).

Suppose N and M/N are left Artinian.

Let $M_1 \supseteq M_2 \supseteq \cdots$ be a descending sequence of left Λ-submodules of M, and for all i let

$M_i'=f(M_i)$.

Since $M_1\cap N\supseteq M_2\cap N\supseteq\cdots$ and $M_1'\supseteq M_2'\supseteq\cdots$ are descending sequences of Λ-modules, by hypotheses there exists an integer i_0 such that $M_i\cap N=M_{i_0}\cap N$, $M_i'=M_{i_0}'$ for all $i\geq i_0$.

Now we consider that if $N_1, N_2(N_1\subseteq N_2)$ are left Λ-submodules of M such that $N_1\cap N=N_2\cap N$ and $f(N_1)=f(N_2)$, then $N_1=N_2$.

If $x\in N_2$, then there exists $y\in N_1$ such that $x-y\in Ker(f)=N$, thus $x-y\in N_2\cap N=N_1\cap N\subseteq N_1$; hence $x\in y+N_1\subseteq N_1$. From $M_{i_0}\supseteq M_{i_0+1}\supseteq\cdots$, it follows that $M_{i_0}=M_{i_0+1}=\cdots$ hence M is an Artinian left Λ-module.

Theorem. 1. *If the left Λ-module M is written as $M=M_1+M_2+\cdots+M_n$, where each M_i is Artinian left Λ-submodules of M, then M is Artinian.*

Proof. It is enough to show the case where $n=2$: $M=M_1+M_2$.
Then $M/M_1=(M_1+M_2)/M_1\cong M_2/(M_1\cap M_2)$.
Since M_2 is Artinian, $M_2/(M_1\cap M_2)$ is Artintan; thus M/M_1 is Artinian. Since M_1 is Artinian, from Lemma 1 M is Artinian.

Lemma 2. *Let M be a left Λ-module. Let $f\in Hom_\Lambda(M, M)$ and i be any positive integer, then*

(1) *If $Im(f)=Im(f\circ f)$, then $Im(f)+Ker(f)=M$*

(2) *If $Ker(f)=Ker(f\circ f)$, then $Im(f)\cap Ker(f)=0$*

(3) *If M be left Artinian, then for every sufficiently large integer i, $M=Im(f^i)+Ker(f^i)$*

(4) *If M is left Noetherian for all sufficiently large integer i, then $Im(f^i)\cap Ker(f^i)=0$*

Proof. (1). Let $Im(f)=Im(f\circ f)$.

For any $x\in M$, there exists $y\in M$ such that $f(x)=(f\circ f)(y)$, thus $f(x-f(y))=0$, hence $x-f(y)\in Ker(f)$ and $x=(x-f(y))+f(y)\in Ker(f)+Im(f)$ therefore $M=Im(f)+Ker(f)$

(2). Let $Ker(f)=Ker(f\circ f)$, and $x\in Im(f)\cap Ker(f)$, then $f(x)=0$ and there exists $y\in M$ such that $x=f(y)$

hence $(f\circ f)(y)=0$.

And so $y\in Ker(f\circ f)=Ker(f)$ implies that $x=f(y)=0$

(3). If M is left Artinian, for all sufficiently large integer i

$$Im(f^i)=Im(f^{2i})$$

Letting f^i instead of f in (1), we have

$$M=Im(f^i)+Ker(f^i).$$

(4). Let M be left Noetherian. For all sufficiently large integer i

$$Ker(f^i)=Ker(f^{2i})$$

Letting f^i instead of f in (2), $Im(f^i)\cap Ker(f^i)=0$

Lemma 3. *If M is a left Artinian Λ-module, then every monomorphism $f\in Hom_\Lambda(M, M)$ is an epimorphism.*

If M is a left Noetherian Λ-module, then every epimorphism $f\in Hom_\Lambda(M, M)$ is a monomorphism.

Proof. Let f be a monomorphism and let M be Artinian.

Then there is an integer i such that $Im(f^i)=M$, since $Ker(f^i)=Ker(f)=0$. From $M\supseteq Im(f)\supseteq Im(f\circ f)\supseteq\cdots\cdots\supseteq Im(f^i)=M$, that is $M=Im(f)$.

Therefore f is an epimorphism.

Let f be an epimorphism and let M be Noetherian, then there exists an integer i such that $Ker(f^i)=0$, since $Im(f^i)=Im(f)'=M$.

From $0\subseteq Ker(f)\subseteq Ker(f\circ f)\subseteq\cdots\cdots\subseteq Ker(f^i)=0$

That is $Ker(f)=0$. Hence f is a monomorphism.

Theorem 2. *Let M be a non-zero indecomposable Λ-module of finite length, and let I be the set of the nonivertible elements of the ring $Hom_\Lambda(M,M)$, then I is the maximal two-sided ideal of $Hom_\Lambda(M,M)$.*

Proof. M has finite length, so M is Artinian and Noetherian.
Let $f\in I$, then from Lemma 3, f is not a monomorphism and not an epimorphism.
From Lemma 2 there exists a sufficiently large integer i such that $M=Im(f^i)+Ker(f^i)$ and $Im(f^i)\cap Ker(f^i)=0$ that is $M=Im(f^i)\oplus Ker(f^i)$. But M is indecomposable,

$$Im(f^i)=0 \text{ or } Ker(f^i)=0$$

If $Ker(f^i)=0$, then $Ker(f)=0$ and so f is monomorphism.
It is impossible, hence

$$Im(f^i)=0, \text{ that is } f^i=0.$$

For any $f,g\in I$, $f+g\in I$. If there exists h such that

$$h\circ(f+g)=1_M, \text{ then } h\circ f+h\circ g=1_M, \text{ and } h\circ f\in I,\ h\circ g\in I.$$

Thus there exists a sufficiently large integer i such that

$$(h\circ f)^i=0,\ (h\circ g)^i=0.$$

Then $1_M=1_M^{2i}=(h\circ f+h\circ g)^{2i}=0$

$$(since\ (h\circ f)\circ(h\circ g)=(1-h\circ g)\circ(h\circ g)=h\circ g-(h\circ g)^2$$
$$=(h\circ g)\circ(1-h\circ g)=(h\circ g)\circ(h\circ f), \text{ and } (h\circ g)^i=(h\circ f)^i=0)$$

For any $f\in I$, $g\in Hom_\Lambda(M,M)$.

$f\circ g\in I$, $g\circ f\in I$. (otherwise $h\circ(g\circ f)=1$, $(h\circ g)\circ f=1$, hence f is an epimorphism, impossible)

Thus I is the two-sided ideal. If J is the ideal contining I properly, then J has an invertible element of $Hom_\Lambda(M,M)$, and so $J=Hom_\Lambda(M,M)$, Therefore I is maximal.

References

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