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On the Vector Norms in Numerical Analysis

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1. Introduction.

In dealing with vectors, matrices, and functions, the problem arises of measuring their size in some convenient form. This is usually done by means of a norm, a nonnegative real-valued function with properties which generalize the usual Euclidean concept of length. When analyzing approximation methods we often need to compare solution or to measure the difference between various answers. In the terminology of linear space we must find the distance between two points in the space. Thus we want to generalize the notions of distance and vector length. The introduction of a norm provides this generalization.

2. Definitions.

Let V be a vector space over F, the field of real or complex numbers.

Let $R^1_+=\{x|x\in R^1\wedge x\geq 0\}$. A function $N: V\longrightarrow R^1_+$ is said to be a *norm on* V if

(i) $N(X)=0\Rightarrow X=0$,

(ii) $\forall X\in V\wedge\forall\alpha\in F,\ N(\alpha X)=|\alpha|N(X)$,

(iii) $N(X+Y)\leq N(X)+N(Y)$,

$\forall X, Y\in V$. Notation: For $N(X)$ we may write $\|X\|$. The set $V=\langle A,\oplus,\odot,F,N\rangle$ or $\langle A,\oplus,\odot,F,\|\ \|\rangle$ will be called a *normed vector space.*

Comment: If $V=\langle A,+,\cdot,F,\|\ \|\rangle$ is a *normed linear vector space* and defining the *distance from* X *to* Y as $\rho(X,Y)=\|X-Y\|$, then the set $\langle A,\rho\rangle$ is a *metric space.* Given a metric space $\langle A,\rho\rangle$, it is not always possible to generate an equivalent normed vector space V. Therefore we say a metric is more general than a norm.

Let $V=\langle A,+,\cdot,F,\|\ \|\rangle$ be a finite-diemensional vector space and let $e^{(1)}, e^{(2)},\cdots e^{(n)}$ be a basis for V. Let $X=\sum_{i=1}^{n}\alpha_i e^{(i)}\in V$. Usually, for $\forall 1\leq p<\infty$, we define the *p-norm* by

$$\|X\|_p=\Big(\sum_{i=1}^{n}|x_i|^p\Big)^{1/p},\quad X\in C^n.$$

Specially, with $p=1$ we have the *absolute norm* $\|X\|_1=\sum_{i=1}^{n}|x_i|$; with $p=2$ we have the *Euclidean norm* $\|X\|_2=(\sum_{i=1}^{n}|x_i|^2)^{1/2}$; and the *maximum norm* $\|X\|_\infty=\max_{1\leq i\leq n}|x_i|$ comes from the fact that $\| X \|_p \longrightarrow \| X \|_\infty$ as $p \to \infty$. This is called variously *infinity norm, Chebyshev norm*, and *uniform norm*. Generally the *weighted p-norm* is defined by $\| X \|_{p,w} = (\sum_{i=1}^{n} w_i | x_i |^p)^{1/p}$, $X \in C^n$, where $w = (w_1, w_2, \cdots w_n)^T$ is a vector of fixed positive weights w_i $i=1, 2, n \cdots$, and p is some number, $1 \le p < \infty$.

3. Theorems.

Theorem 3.1 *Let $N(X)$ be a norm on C^n (or R^n). Then $N(X)$ is a continuous function of the components $x_1, x_2, \cdots x_n$ of X.*

Proof. We want to show that if $x_i \approx y_i$ for $i=1, 2, n$, then $N(X) \approx N(Y)$. Using the *reverse triangle inequality*, $| N(X) - N(Y) | \le N(X-Y)$, $X, Y \in C^n$.
From the definition of the standard basis $\{e^{(1)}, e^{(2)}, \cdots e^{(n)}\}$ for C^n

$$X - Y = \sum_{i=1}^{n} (x_i - y_i) e^{(i)}$$

$$N(X-Y) \le \sum_{i=1}^{n} | x_i - y_i | N(e^{(i)}) \le \| X - Y \|_\infty \sum_{i=1}^{n} N(e^{(i)})$$

$$| N(X) - N(Y) | \le c \| X - Y \|_\infty, \quad c = \sum_{i=1}^{n} N(e^{(i)}).$$

By this theorem we can prove the equivalence of norms: If N and M be norms on C^n, then there are constants $c_1, c_2 > 0$ for which $c_1 M(X) \le N(X) \le c_2 M(X)$ for $\forall X \in V$.

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