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On the Stirling Numbers of the Second Kind.

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1. Introduction.

James Stirling (1692~1770) introduced the number $S_{(n,k)}$ (denoted in this paper) in order to express a function $f(x)$ in the form: $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\sum_{k=1}^{n}S_{(n,k)}[x]_k$, where $[x]_k$ is the *falling factorials*. In combinatorics $S_{(n,k)}$ is the number of ways to partion of an n-set into k nonempty disjoint subsets. Now it will be shown that the two things are equivalent by using the *sieve formula*: $E(0)=\sum_{i=0}^{t}(-1)^iW(i)$, where $E(m)$ is the sum of *weights* of all elements of S which possess exactly m of t properties, and the *binomial inversion*: $v_n=\sum_{k=0}^{n}\binom{n}{k}\Longleftrightarrow u_n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}v_k$, where $u_0,\cdots,u_n$, $v_0,\cdots,v_n$ be reals.

2. Generating function.

Definition 2.1. The number $S_{(n,k)}$ of k-partions is called *Stirling numbers of the 2nd kind.* Hence $S_{(n,k)}>0$ for $1\le k\le n$ and $S_{(n,k)}=0$ if $1\le n<k$. With the convention $S_{(0,0)}=1$ and $S_{(0,k)}=0$ for $k\in \mathrm{N}$. In other words, it is also the number of distributions of n distinct balls into k indistinguishable boxes such that no box is empty.

Theorem 2.2. *For* $\forall n,k\in \mathrm{N}_0$, $x^n=\sum_{k=0}^{n}S_{(n,k)}[x]_k$, *where* $S_{(0,0)}=1$, $S_{(n,0)}=0$ *for* $n>0$ (1)
This horizontal generating function is often taken as the definition of the $S_{(n,k)}$.

Proof. Since any mapping has a unique image we obtain the partition $\mathrm{Map}(N,R)=\bigcup_{B\subset R}\mathrm{Sur}(N,R)$, where $\mathrm{Map}(N,R)$: mappings $N\longrightarrow R$, and $\mathrm{Sur}(N,R)$: surjective mappings $N\longrightarrow R$, and thus $r^n=\sum_{B\subset R}|B|!S_{(n,|B|)}=\sum_{k=0}^{n}\binom{r}{k}\ k!\ S_{(nk)}=\sum_{k=0}^{n}S_{(n,k)}[r]_k$. By replacing r with x, we get (1).

Proof. By using (2) identify the coefficients of $t^n/n!$:

$$\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}=e^{tx}=\{1+(e^t-1)\}^x=\sum_{k=0}^{\infty}\binom{x}{k}(e^t-1)^k$$
$$=\sum_{k=0}^{\infty}[x]_k\frac{(e^t-1)^k}{k!}=\sum_{k=0}^{\infty}[x]_k\frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(e^t)^{k-j}$$
$$=\sum_{k=0}^{\infty}[x]_k\frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{n=0}^{\infty}\frac{(k-j)^nt^n}{n!}$$
$$=\sum_{k=0}^{\infty}[x]_k\sum_{n=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\ (k-j)^n\frac{t^n}{n!}$$

$$=\sum_{k=0}^{\infty}[x]_k\sum_{n=0}^{\infty}S_{(n,k)}\frac{t^n}{n!}\quad (by\ (2))$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}[x]_kS_{(n,k)}\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}[x]_kS_{(n,k)}\frac{t^n}{n!}.$$

Hence (1) holds.

Theorem 2.3 $S_{(n,k)}=\frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$ (2)

$$=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n \quad (3)$$

Proof. (Combinatorial approach)

$|\mathrm{Sur}(N,K)|$ means the number of the ways in the case of k distinct boxes in Definition.

This is equivalent to the number of words of length n over an alphabet of k letters, such that each letter appears in each word at least once. We shall use the sieve formula. Let P_i, $1\le i\le k$, be the property that Ci does not appear in a word.

Thus, $W(P_i)=(k-1)^n$ and for $1\le l\le k$, $W(P_{i_1}, P_{i_2}, \cdots P_{i_l})=(k-l)^n$.

Thus, $W(m)=\binom{k}{m}(k-m)^n$, and by the sieve formula, $|\mathrm{Sur}(N,K)|=\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$.

Since $|\mathrm{Sur}(N,K)|=k!S_{(n,k)}$, we get (2). And by the binomial inversion, we get (3).

Proof 2. (Analytical approach) .

By means of Harriot-Briggs formula we can write x^n as a series in the functions $[x]_k$

$$x^n=\sum_{k=1}^{n}\frac{1}{k!}[x]_k\Delta^kx^n|^{x=0}.$$

Then by (1) $S_{(n,k)}=\frac{1}{k!}\Delta^kx^n|^{x=0}$.

To evaluate these numbers let us following Jordan, symbolically expand $\Delta^k=(E-I)^k$ in to a series: $\Delta^k=\sum_{i=0}^{k}(-1)^i\binom{k}{i}E^{k-i}$.

If we then apply these operators to x^n, we find (2):

$$\left[\Delta^k\frac{x^n}{k!}\right]_{x=0}=S_{(n,k)}=\frac{1}{k!}\sum_{i=0}^{k}(-1)^i(k-i)^n.$$

Similarly if we apply $\Delta^k=(-1)(I-E)^k$ to x^n, we get (3).

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