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Some Properties of Linear Functionals on Two-Norm Spaces

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The purpose of this note is to show some basic properties of γ-linear functionals on two-norm spaces and Saks spaces, and apply the result to the theory of Banach Spaces.

We begin with some definitions given in [1], [2], and [3].

A Frechet norm $|\ |$ on a linear set X is a real valued nonnegative function with the following properties:

(1) $|x|=0$ if and only if $x=0$,

(2) $|x+y|\leq|x|+|y|$ for all x,y in X,

(3) if $\{a_n\}$ is a sequence of real numbers converging to a real number a and $\{x_n\}$ is a sequence of points of X with $|x_n-x|\longrightarrow 0$, then $|a_nx_n-ax|\longrightarrow 0$.

It is called a B-norm if the condition (3) is replaced by

(4) $|ax|=|a||x|$ where a is any real number and x is any element of X.

Let $|\ |_1$ and $|\ |_2$ be two norms (B or F) defined on X. We define $|\ |_1\geqq|\ |_2$ if $|x_n|_1\longrightarrow 0$ implies $|x_n|_2\longrightarrow 0$.

When $|\ |_2\geqq|\ |_1$ and $|\ |_2\geqq|\ |_1$, we say that $|\ |_1$ is equivalent to $|\ |_2$ and write $|\ |_1\sim|\ |_2$.

A two norm space is a linear set X with two norms, a B-norm $|\ |_1$ and an F-norm $|\ |_2$. A sequence x_n of points in a two norm space $(X, |\ |_1, |\ |_2)$ is said to be γ-convergent to x in X, written $x_n\longrightarrow x$, if

$\lim\sup_n |x_n|_1<\infty$ and $\lim_n |x_n-x|_2=0$.

A sequence $\{x_n\}$ in a two norm space is said to be γ-Cauchy if $(x_{p_n}-x_{q_n})\rightarrow 0$ as $p_n, q_n\rightarrow\infty$.

A two norm space $X_s=(X, |\ |_1, |\ |_2)$ is called γ-complete if for every γ-Cauchy sequence $\{x_n\}$ in X_s there exists an x in X_s such that $x_n\longrightarrow x$.

A γ-linear functional f on two norm space is a real valued function on X_s such that

(1) $f(ax+by)=af(x)+bf(y)$, for every real numbers a, b and any x,y in X_s,

(2) if $x_n\longrightarrow x$, then $f(x_n)\longrightarrow f(x)$.

The set of all γ-linear functionals on X_S will be denoted by X_s^*. It is easy to see that X_s^* is a linear set.

Let X be a linear set and suppose that $|\ |_1$ is a B-norm, and $|\ |^*$ is an F-norm on X.

Let $X_s=\{x\in X: |x|_1\leq 1\}$ and define $d(x,y)=|x-y|^*$ for x,y in X_s.

Then d is a metric on X_s and the metric space (X_s,d) will be called a Saks set. If (X_s,d) is complete, it will be called a Saks space. We shall denote (X_s,d) by $(X, |\ |_1, |\ |^*)$. A linear functional L_X on $(X_s^*, |\ |_1^*)$ is defined by $L_x(f)=f(x)$ for each f in X_s^*.

If we work in the setting of Saks sets, we can use category arguments. A disadvantage is that a Saks set is not a linear set while a two norm space is a linear set.

W. Orlicz [4] has shown the following

Lemma. *Let $(X_s,d)=(X, |\ |_1, |\ |_2)$ be a Saks space. Then the follcwing are equivalent:*

(1) *$|\ |_1$ is equivalent to $|\ |_2$ on X,*

(2) *$|\ |_2\geq|\ |_1$ on X and $(X, |\ |_1)$ is a Banach space,*

(3) *$|\ |_1\geq|\ |_2$ on X and $(X, |\ |_2)$ is a Frechet Space.*

Using this lemma we can show the following

Theorem. *Let $(X_s,d)=(X, |\ |_1, |\ |_2)$ be a Saks space. Then the following are equivalent:*

(1) *$|\ |_1$ is equivalent to $|\ |_2$ on X,*

(2) *$|\ |_2\geq|\ |_1$ on X and $|\ |_1\geq|\ |_2$ on a linear set A with $A\cap X_s$ dense in X_s with respect to $|\ |_1$.*

Proof. By the lemma it is sufficient to show that $(X, |\ |_1)$ is a Banach space. Let $\{x_n\}$ be an arbitrary Cauchy sequence in $(X, |\ |_1)$. Then by the definition of Saks space there exists a non-zero constant c such that $|x_n|_1\leq c$. Let $y_n=x_n/c$. Then clearly y_n belongs to X_s for every n.

Since $A\cap X_s$ is dense in X_s, for each n, there exists a sequence in $A\cap X_s$ $\{z_{k,n}\}$ such that for each k, $|y_n-z_{k,n}|_1<1/n$ and

$$\lim_k |y_n-z_{k,n}|_1=0.$$

Now we show that the sequence $\{z_{n,n}\}$ is a Cauchy in $(X, |\ |_1)$. By appling the triangle inequality, we have

$$|z_{n,n}-z_{m,m}|_1\leq|z_{n,n}-y_n|_1+|y_n-y_m|_1+|y_m-z_{m,m}|_1.$$

As n and m tend to infinity each of the terms on the right approaches zero. Hence $\{z_{n,n}\}$ is a Cauchy sequence in $(X, |\ |_1)$.

We may assume that $|z_{n,n}-z_{m,m}|_1<r<1$. Since $z_{n,n}-z_{m,m}$ belongs to $A\cap X_s$, the condition $|\ |_1\geq|\ |_2$ on A implies that $|z_{n,n}-z_{m,m}|_2$ approaches zero as n,m tend to infinity. Thus $\{z_{n,n}\}$ is a Cauchy sequence in (X_s,d). The space (X_s,d) being complete, $\{z_{n,n}\}$ converges to an element x in X_s. On the other hand, the condition $|\ |_2\geq|\ |_1$ on X implies that $\{z_{n,n}\}$ tends to x in $|\ |_1$ as n tends to infinity. Then, the inequality

$$|y_n-x|_1\leq|y_n-z_{n,n}|_1+|z_{n,n}-x|_1$$

shows that y_n approaches x in $|\ |_1$ as n tends to infinity and $(X, |\ |_1)$ is complete.

W. Orlicz and V. Ptak ([5], p. 63) have defined a set Y_0 in $(X, |\ |_X)^*$ to have property (T) if for each non-zero x in X, there is a y in Y_0 shuch that $y(x)\neq 0$.

Let V_0 be a subspace of $(X, |\ |_X)^*$. A subset V of V_0 is called an 0-basis for V_0 if

(1) norm of each functionals in V is at most one, and

(2) V_0 is the closure with respect to $|\ |_X^*$ of the linear envelope of V.

If Y_0 in $(X, |\,|_X^*)$ has property (T), and V is an 0-basis for Y_0, a B-norm can be defined on X by

$$|x|_0^* = \sup_{y \in V} |y(x)|.$$

If Y_0 is separable and $\{f_n\}$ is dense in the unit ball, $n^{-1} f_n$ is a compact 0-basis for Y_0. If V is a compact 0-basis for Y_0, we define $(X_{s,n}, d) = (X, \;|\;|_X, \;|\;|_0^*)$.

If $(X, \;|\;|_X)^*$ is separable, X^* has a compact 0-basis V. The space of γ-linear functionals on $(X_{s,n}, d)$ will be denoted by $X_{s,n}^*$.

Now we are ready to state the following

Lemma. $X^* = X_{s,n}^*$.

Proof. The inclusion $X_{s,n}^* \subset X^*$ is clear from the fact that $|\;|_0^* \leq |\;|_X$. Let y belong to V. Then f belongs to $X_{s,n}^*$ by the definition of $|\;|_0^*$. The space $(X_{s,n}^*, \;|\;|_{X^*})$ is a Banach space ([5], p. 57). Thus $X_{s,n}^*$ contains the closure with respect to $|\;|_X^*$ of the linear envelope of V, which is X^*. Hence $X^* = X_{s,n}^*$.

Theorem. *In $(X_{s,n}, d)$, γ-weak convergence is equivalent to $|\;|_0^*$ convergence.*

Proof. Let x_n converge to x weakly in $(X_{s,n}, d)$. By the lemma $X_{s,r}^* = X^*$. Therefore, $L_{x_n}(f) = f(x_n)$ approaches $f(x) = L_x(f)$ for each f in $X_{s,n}^* = X^*$.

Since V is compact, by Gelfand's Theorem ([6], p. 299), L_{x_n} tends to L_x uniformly on V_0. Hence, for any given $\varepsilon > 0$, there is a natural number N such that

$$|L_{x_n}(f) - L(f)| = |f(x_n) - f(x)| < \varepsilon$$

for $n > N$, for all f in V, Thus

$$|x_n - x|_0^* = \sup_{f \in V} |L_{x_n}(f) - L_x(f)| < \varepsilon.$$

Hence $|x_n - x|_0^*$ approaches zero as n tends to infinity. This completes the proof of the theorem.

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