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Operator Spaces which are Non—F—Spaces

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1. Introduction

In this note some theorem about operator space in which the image of a set under u is not always a closed set, are shown. The operator u on a set X is a mapping, which assigns a set $uA\subset X$ to every $A\subset X$ and satisfies the following axioms: $u\phi=\phi$, $u(X)=X$, $A\subset uA$, $u(A\cup B)=(uA)\cup(uB)$. The condition $u(uA)=uA$ called axiom F, is not required in general thus we distinguish among F-spaces (i. e., spaces satisfying F-axiom) and non-F-spaces.

In non-F-spaces neighborhood of point $x\in X$ and interior of sets are defined as follows: a set U is a neighborhood of a point $x\in X$ if $x\notin u(X-U)$;

$$int\ A=\{x\in A : A \text{ is a nbd of } x\}$$

We can induce a topology from the given operator space and show that initial and final operators in the some of Bourbaki of operators in case of monosources and episinks respectively exist.

2. Definitions and theorems.

We denote by $cl[K]$ the closure of a subset K of a topological space and by $int[K]$ the interior of a subset K

Definition 2. 1 A point is in the *θ-closure* of a subset K of a space if each open subset V of the space with

$$x\in V \text{ satisfies } K\cap cl[V]\neq\phi$$

In this case we write $x\in\theta\text{-}cl[K]$

Lemma 2. 1 *If X is a regular space, then every subset A of X,*

$$cl(A)=\theta-cl(A)$$

Proof. It is obvious that $cl\ (A)\subset\theta-cl(A)$.

To prove the converse, let $x\in\theta-cl\ A$.

Then for every open subset V with $x\in V$, $cl(V)\cap A\neq\phi$, Since X is regular, for every open set W with $x\in W$, there is an open set V with $x\in V$ such that $cl[V]\subset W$, Hence $A\cap W\neq\phi$, Thus $x\in cl(A)$

Lemma 2. 2. *For every open set U of a space X,*

$\theta-cl[U]=cl[U]$.

Proof. Let $x\notin cl[U]$, There is an open subset V with $x\in V$ such that $V\cap U=\phi$ Thus $V\subset U^c$, Since U^c is closed, $cl(V)\subset U^c$. i.e., $clV\cap U=\phi$ Thus $x\notin\theta-cl[U]$.

Next two theorems show the examples of operators space.

Theorem 2.3. *Let A, B be subsets of a topological space X.*

(1) $\theta-cl[\phi]=\phi$, $\theta-cl[X]=X$

(2) $A\subset\theta-cl[A]$

(3) $\theta-cl[A\cup B]=\theta-cl[A]\cup\theta-cl[B]$

(4) *In general,* $\theta-cl[\theta-cl[A]]\neq\theta-cl[A]$

Proof. (1) and (2) are obvious by the definition. It is sufficient to show (3).

Since $\theta-cl[A]\subset\theta-cl[A\cup B]$, $\theta-cl[B]\subset\theta-cl[A\cup B]$,

$\theta-cl[A]\cup\theta-cl[B]\subset\theta-cl[A\cup B]$.

Let $x\in\theta-cl[A\cup B]$. Then for every open set U with $x\in U$, $cl[U]\cap(A\cup B)\neq\phi$.

Thus $cl[U]\cap A\neq\phi$ or $cl[U]\cap B\neq\phi$. i.e., $x\in\theta-cl[A]$ or $x\in\theta-cl[B]$

Hence $\theta-cl[A\cup B]\subset\theta-cl[A]\cup\theta-cl[B]$

Theorem 2.4 *For any convergence space (X, C), the following holds* [2].

(1) $A\subseteq\bar{A}$

(2) $\bar{\phi}=\phi$

(3) $\overline{A\cup B}=\bar{A}\cup\bar{B}$ *are hence* $A\subseteq B$ *implies* $\bar{A}\subset\bar{B}$

(4) *In general,* $A\neq\bar{A}$

Definition 2.2. A subset A of an operator space (X, u) is said to be *closed* in (X, u) if $uA=A$.

Lemma 2.5. *For any operator space (X, u), $\mathcal{F}=\{A\mid A$ is closed in $(X, u)\}$ is the family of closed set for some topology on X.*

Proof. (i) $uX=X$, $u\phi=\phi$ *i.e.,* $X, \phi\in\mathcal{F}$

(ii) If $A, B\in\mathcal{F}$, then $u(A\cup B)=u(A\cup B)=u(A)\cup u(B)=A\cup B$. Thus $A\cup B\in\mathcal{F}$.

(iii) If $(A_i)_i\in I\subset\mathcal{F}$, then $\cap A_i\subseteq u(\cap A_i)\subseteq\cap uA_i=\cap A_i$. Thus $\cap A_i\in\mathcal{F}$

Definition 2.3 Let (X, u) be an operator space, then the topology defined by $\{A\mid A^c$ is closed in $(X, u)\}$ is called the topology on X induced by the operator u

Remark We note that for a operator space (X, u) and a subset A of X, uA is not necessarily the clousure of A with respect to the induced topology by u

Theorem 2.6. '*Let (X, u) be an operator space. Then a subset A of X is open in the induced topology on X by u iff $intA=A$.*

Proof. Suppose $intA=A$, Let $x\in A=intA$, i.e., $x\notin A^c$ Since A is a neighborhood of x, $x\notin u(A^c)$ Thus $u(A^c)\subset A^c$. Hence $u(A^c)=A^c$, i.e., A is open in the induced topology on X by u.

Conversely, suppose $u(A^c)=A^c$. Let $x\in A$. Then $x\notin A^c=u(A^c)$. Since $x\notin u(A^c)$, A is a neighborhood of x, Thus $x\in intA$, i.e., $int\ A=A$.

Definition 2.4. Let (X, u) and (Y, u') be the operator spaces. A map $f: X\longrightarrow Y$ is continuous if for every subset A of X, $f(uA)\subset u'f(A)$.

Theorem 2.7. *For every operator space* (X, u), *the identity map* $I_x : (X, u) \longrightarrow (X, u)$ *is continuous. And if* $f : (X, u) \longrightarrow (Y, u')$ *is continuous and* $g : (Y, u') \longrightarrow (Z, u'')$ *is continuous, then* $gf : (X, u) \longrightarrow (Z, u'')$ *is also continuous.*

Proof. The proof of the first part of the above theorem is obvious by the definition and that of the second part is obvious since for every subset A of X,

$$(g \cdot f)(uA) \subset gu'(f(A)) \subset u''(gf(A)).$$

Thus gf is continuous.

Remark. By the above theorem, the class of all operator space and continuous maps form a category which will be denoted by *Operator*.

Theorem 2.8. *Let* $((X_i, u_i))_{i \in I}$ *be a family in Operator and* $f_i : X \longrightarrow X_i$ *a map for each* $i \in I$. *Then there is an operator* u *on* X *such that a monosouce* $(f_i : (X, u) \longrightarrow (X_i, u_i))_{i \in I}$ *is initial in the sense of Bourbaki, i.e., for each* $i \in I$, f_i *is continuous, and for any operator space* (Y, u'), *a map* $g : (Y, u') \longrightarrow (X, u)$ *is continuous iff for each* $i \in I$, $f_i g : (Y, u') \longrightarrow (X_i, u_i)$ *is continuous.*

Proof. Define u as follows: For each i and for every subset A *of* X, let $f_i(uA) = u_i f_i(A)$. Let's first show that u is an operator satisfying the conditions.

(i) For each $i \in I$,

$$f_i(u\phi) = u_i f_i(\phi) = u_i \phi = \phi = f_i(\phi).$$

Thus $u\phi = \phi$.

For each $i \in I$,

$$f_i(uX) = u_i f_i(X) \supset f_i(X)$$

Thus $uX \supset X$. Hence $uX = X$.

(ii) For each $i \in I$,

$$f_i(A) \subset u_i f_i(A) = f_i(uA).$$

Thus $A \subset uA$.

(iii) For each $i \in I$,

$$\begin{aligned} f_i u(A \cup B) &= u_i f_i(A \cup B) \\ &= u_i(f_i(A) \cup f_i(B)) \\ &= u_i f_i(A) \cup u_i f_i(B) \\ &= f_i(uA) \cup f_i(uB) \\ &= f_i((uA) \cup (uB)). \end{aligned}$$

Thus $u(A \cup B) = uA \cup uB$.

Hence u is an operator on X satisfying the condition.

Obviously for each $i \in I$, f_i is continuous.

Suppose for each $i \in I$, $f_i g : (Y, u') \longrightarrow (X_i, u_i)$ is continuous.

Then $f_i g(u'A) \subset u_i f_i g(A) = f_i(ug(A))$

Since f is monosource, $g(u'A) \subset ug(A)$

Hence g is continuous. This completes the proof.

Definition 2.5. The operator u on X defined in the above theorem is called the *initial monosource operator* with respect to $(f_i)_{i \in I}$.

Definition 2. 6. For a operator space (X, u) and a subset A of X, the initial monosource operator u_A with respect to the natural embedding $A \longrightarrow X$ is called the *relation operator of u on A*, and (A, uA) is called a subspace of (X, u).

Definition 2. 7. Let $((X, u_i))_{i \in I}$ be a family of operator spaces indexed by a set I. Then the initial monosource operator Πu_i on ΠX_i with respect to projections is called the product operator and $(\Pi X_i, \Pi u_i)$ is called the product operator space of the family. The following theorem is immediate by the above theorem.

Theorem 2. 9. *a map $f: (X, u) \longrightarrow (\Pi X_i, \Pi u_i)$ is continuous. iff for each $i \in I$, $P_{r_i} f: (X, u) \longrightarrow (X_i, u_i)$ is continuous. Hence $(\Pi X_i, \Pi u_i)$ is a categorical product of $((X_i, u_i))_{i \in I}$ in Operator.*

Theorem 2. 10. *Let $((X_i, u_i))_{i \in I}$ be a family of operator spaces and for each $i \in I$, $f_i: X_i \longrightarrow X$ a map. Then there is an operator u such that an episink $(f_i: (X_i, u_i) \longrightarrow (X, u))_{i \in I}$ is final in the sense of Bourbaki.*

Proof. Define u as follows: For each $i \in I$ and for every subset B of X_i, let $f_i(u_i\, B) = uf_i(B)$.

To show that u is an operator satisfying the conditions. Since $(f_i: X_i \longrightarrow X)_{i \in I}$ is an episink for every subset C of X, there is an $i \in I$ and $C' \subset X_i$ such that $f_i(C') = C$. Thus $uC = uf_i(C') = f_i(uC') \supset f(C') = C$, It is obvious to show that $u\phi = \phi$ and $uX = X$. It remains to show that $u(A \cup B) = (uA) \cup (uB)$. Since $(f_i: X_i \longrightarrow X)_{i \in I}$ is an episink, for subsets A, B of X, there is an $i \in I$ and subsets A', B' of X_i such that $f_i(A') = A$ and $f_i(B') = B$.

$$
\begin{aligned}
\text{Thus } u(A \cup B) &= u(f_i(A') \cup f_i(B')) \\
&= uf_i(A' \cup B') \\
&= f_i(u_i(A' \cup B')) = f_i(u_i A' \cup u_i B') \\
&= f_i(u_i A') \cup f_i(u_i B') = uf_i(A') \cup uf_i(B') \\
&= uA \cup uB.
\end{aligned}
$$

Obviously for each $i \in I$, f_i is continuous. Suppose that for an operator space (Y, u') and a map $g: X \longrightarrow Y$, $gf_i: (X_i, u_i) \longrightarrow (Y, u')$ is continuous. Then for every subset B of X, there is an $i \in I$ and $A \subset X_i$ such that $f_i(A) = B$. Thus $g(uB) = g(u(f_i(A))) = gf_i(u_i A) \subset u'gf_i(A) = u'g(B)$. Hence g is continuous. This completes proof.

Definition 2. 8. The operator u defined in the above theorem is called the *final operator* on X with respect to episink $(f_i)_i \in I$

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