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On the Cauchy Probability Density Functions

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If $X_1, X_2, \ldots, X_n$ are independent random variables each having the Cauchy probability density function, then $(X_1+X_2+\cdots+X_n)/n$ also has the Cauchy probability density function.

Lemma 1. *Let X be a continuous random variable having the probability density function* $f_x(\pi)=\frac{1}{2}e^{-|x|}$, $-\infty<x<\infty$.

Then a) the characteristic function of X is given by $\varphi_x(t)=\frac{1}{1+t^2}$

b) $e^{-|x|}=\int_{-\infty}^{\infty} e^{itx}\frac{1}{\pi(1+t^2)}dt.$

Proof. a) $\varphi_x(t)=Ee^{itx}=\int_{-\infty}^{\infty}\frac{1}{2}e^{-|x|}e^{itx}dx$

$$=\int_{-\infty}^{0}\frac{1}{2}e^{x}e^{itx}dx+\int_{0}^{\infty}\frac{1}{2}e^{-x}e^{itx}dx$$

$$=\frac{1}{2}\int_{-\infty}^{0}e^{(1+it)x}dx+\frac{1}{2}\int_{0}^{\infty}e^{(-1+it)x}dx$$

$$=\frac{1}{2}\left(\frac{1}{1+it}e^{(1+it)x}\right)_{-\infty}^{0}+\frac{1}{2}\left(\frac{1}{-1+it}e^{(-1+it)x}\right)_{0}^{\infty}$$

$$=\frac{1}{2}\left(\frac{1}{1+it}-\frac{1}{-1+it}\right)=\frac{1}{1+t^2}$$

b) By a) and inversion formula $f_x(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\varphi_x(t)dt$,

$$\frac{1}{2}e^{-|x|}=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\varphi_x(t)dt.$$

$$e^{-|x|}=\int_{-\infty}^{\infty}e^{-itx}\frac{1}{\pi(1+t^2)}dt.$$

Substituing $-x$ for x, $e^{-|x|}=\int_{-\infty}^{\infty}e^{itx}\frac{1}{\pi(1+t^2)}dt.$

Lemma 2. *Let X be a random variable having the Cauchy probability density function* $f_x(x)=\frac{1}{\pi(1+x^2)}$, $-\infty<x<\infty$.

Then the characteristic function of X is given by $\varphi_x(t)=e^{-|t|}$.

Proof. By lemma 1 b), $\int_{-\infty}^{\infty}e^{itx}\frac{1}{\pi(1+t^2)}dt=e^{-|x|}$.

Interchange the role of x and t.

$\int_{-\infty}^{\infty} e^{itx}\frac{1}{\pi(1+x^2)}dx=e^{-|t|}$, which is the desired conclusion.

Theorem. *If $X_1, X_2, \ldots, X_n$ are independent random variables each having the Cauchy probability density function, then $(X_1+X_2+\cdots+X_n)/n$ also has the Cauchy probability density function.*

Proof. By induction, we shall prove only the case $n=2$.

$$\varphi_{X+Y}(t)=Ee^{it(X+Y)}=Ee^{itX}Ee^{itY}=\int_{-\infty}^{\infty} e^{itx}\frac{1}{\pi(1+x^2)}dx\int_{-\infty}^{\infty} e^{itx}\frac{1}{\pi(1+y^2)}dy$$
$$=e^{-|t|}e^{-|x|}=(e^{-|t|})^2,$$

which is resulted by lemma 2.

$$\varphi_{\frac{X+Y}{2}}(t)=Ee^{it\frac{X+Y}{2}}=Ee^{i\frac{t}{2}X}Ee^{i\frac{t}{2}Y}=\varphi_X(\tfrac{t}{2})\,\varphi_Y(\tfrac{t}{2})$$
$$=e^{-\left|\frac{t}{2}\right|}e^{-\left|\frac{t}{2}\right|}=\left(e^{-\left|\frac{t}{2}\right|}\right)^2=e^{-|t|}.$$

By inversion formula,

$$f_{\frac{X+Y}{2}}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}e^{-|t|}dt$$
$$=\frac{1}{2\pi}\int_{-\infty}^{0} e^{-itx}e^{t}dt+\frac{1}{2\pi}\int_{0}^{\infty}e^{-itx}e^{-t}dt$$
$$=\frac{1}{2\pi}\int_{-\infty}^{0} e^{(1-ix)t}\,dt+\frac{1}{2\pi}\int_{0}^{\infty}e^{-(1+ix)t}\,dt$$
$$=\frac{1}{2\pi}\left(\frac{1}{1-ix}e^{(1-ix)t}\right)_{-\infty}^{0}+\frac{1}{2\pi}\left(\frac{1}{-(1+ix)}e^{-(1+ix)t}\right)_{0}^{\infty}$$
$$=\frac{1}{2\pi}\left(\frac{1}{1-ix}+\frac{1}{1+ix}\right)$$
$$=\frac{1}{2\pi}\quad\frac{1}{1-(ix)^2}=\frac{1}{\pi(1+x^2)}.$$

This is the Cauchy probability density function which is desired result.

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