韓國數學教育學會誌 <數學教育>
1980. 12 Vol. XIX No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1980. Vol. XIX No. 1

A note on Continuous Seminorms on Locally Convex Spaces.

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1. Introduction

In this note, we shall study some properties of a basis of continuous seminorms on the locally convex space E. Finally, we shall prove the following facts: If $\mathcal{P}$ is a basis of continuous seminorms on E, then the family of seminorms on E/M $\dot{\mathcal{P}}$ induced by $\mathcal{P}$ is a basis of continuous seminorms of E/M.

2. Preliminaries

In this section, we collect some definitions and known results which we will need in this note.

Definition 2.1 A subset K of a vector space E is *convex* if, whenever K contains two points x and y, K also contains the segment of straight line joining them: If $x, y \in K$ and if α, β are two numbers ≥ 0 such that $\alpha+\beta=1$, then $\alpha x+\beta y \in K$

Definition 2.2 A subset K of a vector space E is said to be *absorbing* if to every $x \in E$, there is a number $C_x > 0$ such that for all $\lambda \in C$ $|\lambda| \leq C_x$, we have $\lambda x \in K$.

Definition 2.3 A subset K of a vector space E is said to be *balanced* if for every $x \in K$ and every $\lambda \in C$, $|\lambda| \leq 1$, we have $\lambda x \in K$

Definition 2.4 A subset T of a topological vector space (TVS) E is called a *barrel* if T has the following four properties:

(1) T is absorbing;

(2) T is balanced;

(3) T is closed;

(4) T is convex.

Definition 2.5 A TVS E is said to be a *locally convex space* if there is a basis of neighborhoods in E consisting of convex sets.

Lemma 2.1 *In a locally convex space E, there is a basis neighborhoods of O consisting of barrels.*

Definition 2.6 A nonnegative function $p : x \longmapsto p(x)$ on a vector space E is called a *seminorm* if p satisfies the following conditions;

(1) p is subadditive, i.e., for all $x, y \in E$, $p(x+y) \leq p(x)+p(y)$

(2) p is positively homogeneous of degree 1, i.e., for all $x \in E$ and all $\lambda \in C$, $p(\lambda x) = |\lambda| p(x)$.

Definition 2.7 Let E be a vector space and p a seminorm on E. Then the sets

$$U_p = \{x \in E | p(x) \leq 1\}, \quad \mathring{U}_p = \{x \in E | p(x) < 1\}$$

will be called, respectively, the *closed and the open unit semiball of p*.

Lemma 2.2 *Let E be a TVS and p a seminorm on E. Then the following statements are equivalent;*

(1) *The open unit semiball of p is an open set;*

(2) *p is continuous at the origin;*

(3) *p is continuous at every point.*

Lemma 2.3 *If p is a continuous seminorm on TVS E, then its closed unit semiball is a barrel.*

Lemma 2.4 *Let E be a TVS and T a barrel in E. Then there exists a unique seminorm p on E such that T is the closed unit semiball of p. The seminorm p is continuous if and only if T is a neighborhood of* 0.

Corollary. *Let E be a locally convex space. The closed unit semiballs of the continuous seminorms on E form a basis of neighborhoods of* 0.

Definition 2.8 A family of continuous seminorms $\mathcal{P}$ on a locally convex space E is said to be a *basis of continuous seminorms on E* if to any continuous seminorm p on E there is a seminorm q beloning to $\mathcal{P}$ and constant $c > 0$ such that, for all $x \in E$,

$$(A) \quad p(x) \leq cq(x)$$

(Remark) Let us denote by U_p (resp. U_q) the closed unit semiball of p (resp. q). Then (A) means $c^{-1} U_q \subset U_p$

3. Some properties

Proposition 3.1 *Let $\mathcal{P}$ be a basis of continuous seminorms on the locally convex space E. Then the sets λU_p, where U_p is the closed unit semiball of p and p varies over $\mathcal{P}$ and λ on the set of numbers>0, form a basis of neighborhoods of* 0. *Conversely, given any family of neighborhoods of* 0, *$\mathcal{B}$, consisting of barrels and such that the set λU, when $U \in \mathcal{B}$ and $\lambda > 0$, form a basis of neighborhoods of* 0 *in E, then the seminorms whose closed unit semiballs are the barrels belonging to $\mathcal{B}$ form a basis of continuous seminorms in E.*

Proof ($\Rightarrow$) By Lemma 2.1., let v be a barrel which is a neighborhood of 0. Then, by Lemma 2.4., there exists a unique continuous seminorm p on E such that $v = U_p$. From Definition 2.9., there is $q \in \mathcal{P}$ and a constant $c > 0$ such that for all $x \in E$, $p(x) \leq cq(x)$

From(Remark),

$$\frac{1}{c} U_q \subset U_p = v, \text{ i.e., } \lambda U_q \subset v, \text{ where } \lambda = \frac{1}{c}$$

Thus $\{\lambda U_q | q \in \mathcal{P}, \lambda > 0\}$ is a basis of neighborhoods of 0.

($\Leftarrow$) Let $\mathcal{P} = \{q | q$ is a continuous seminorm on E whose closed unit semiball $U_q \in \mathcal{B}\}$ and let p be any continuous seminorm on E, Then U_p is a neighborhood of 0. By assumption, there is $U \in \mathcal{B}$ and $\lambda > 0$ such that $\lambda U \subset U_p$ *or* $U \subset \frac{1}{\lambda} U_p$ and we have $q \in \mathcal{P}$ such that $U_q = U$. Thus $\lambda U_q \subset U_p$, i.e., $p(x) \leq \frac{1}{\lambda} q(x)$, for all $x \in E$.

Hence $\mathcal{P}$ is a basis of continuous seminorms on E.

Proposition 3.2. *Let E, F be two locally convex spaces. A linear map $f: E \longrightarrow F$ is continuous if and only if to every continuous seminorm q on F, there is a continuous seminorm p on E such that, for all $X \in E$, $q(f(x)) \leq p(x)$.*

Proof ($\Rightarrow$) Suppose that f is continuous linear map. Then, since q is a continuous seminorm on F, $q \circ f$ is a continuous seminorm on E. Let $p = q \circ f$. Then, for every $x \in E$, $q(f(x)) = p(x)$ or $q(f(x)) \leq p(x)$.

($\Leftarrow$) Suppose that for any continuous seminorm q on F, there is a continuous seminorm p on E such that, for each $X \in E$, $q(f(x)) \leq p(x)$. Now let U be a basic neighborhood barrel. Then there is a continuous seminorm q such that $U = U_q$.

On the other hand, since there is a continuous seminorm p on E such that, for all $x \in E$, $q(f(x)) \leq p(x)$ and $q \circ f$ is a continuous seminorm on E,

$$U_p \subset U_{q \circ f} = f^{-1}(q^{-1}([0, 1])) = f^{-1}(U_q) \textit{ i.e., } U_p \subset f^{-1}(U_q).$$

Thus $f^{-1}(U_q)$ is a neighborhood of 0 in E.

Proposition 3.3 *Let E be a locally convex space. Let $\mathcal{P}$ be a basis of continuous seminorms on E. A filter $\mathcal{F}$ on E converges to a point x if and only if to every $\varepsilon > 0$ and to every seminorm $p \in \mathcal{P}$ there is a subset M of E belonging to $\mathcal{F}$ such that, for all $y \in M$, $p(x-y) < \varepsilon$*

Proof ($\Rightarrow$) Let $p \in \mathcal{P}$. Then for every $\varepsilon > 0$, $\frac{1}{\varepsilon} p$ is a continuous seminorm on E. Let $V = x + \mathring{U}_{\frac{p}{\varepsilon}}$. Since $F \longrightarrow x$, $\exists M \in \mathcal{F}$ such that $M \subset V = x + \mathring{U}_{\frac{p}{\varepsilon}}$. Hence for every $y \in M$, we have $y \in x + \mathring{U}_{\frac{p}{\varepsilon}}$. Then $X - Y \in \mathring{U}_{\frac{p}{\varepsilon}}$, i.e., $\frac{1}{\varepsilon} p(x-y) < 1$. Thus $p(x-y) < \varepsilon$.

($\Leftarrow$) Let $x + U$ be a neighborhood of x. (where U is a barrel.) Then there is a continuous seminorm p on E such that $U_p = U$.

Since $\mathcal{P}$ is a basis of continuous seminorms on E, there is $q \in \mathcal{P}$ and a constant $c > 0$ such that $\frac{1}{c} p(x) \leq q(x)$, *i.e.*, $\frac{1}{c} U_q \subset U_p = U$. Let $\varepsilon = \frac{1}{c}$.

Then there is $M \in \mathcal{F}$ such that, for all $y \in M$, $q(y-x) < \varepsilon$.

Thus $y - x \in \frac{1}{c} U_q \subset U$, i.e., $y \in x + U$ or $M \subset x + U$.

Hence $x + U \in \mathcal{F}$ or $\mathcal{F} \longrightarrow x$.

Proposition 3.4 *Let E be a locally convex space, and M a linear subspace of E. Let ϕ be the canonical mapping of E onto E/M. If $\mathcal{P}$ is a basis of continuous seminorms on E, let us denote by $\dot{\mathcal{P}}$ the family of seminorms on E/M consisting of the seminorms*

$$E/M \ni \dot{x} \longrightarrow \dot{p}(x) = \inf p(x). \quad \phi(x) = \dot{x}.$$

Then $\mathcal{P}$ is a basis of continuous seminorms of E/M.

Proof (i) For all $\dot{x}, \dot{y} \in E/M$,

$$\dot{p}(\dot{x} + \dot{y}) = \dot{p}(x + y + M)$$

$$=\inf p(x+m+y+m') \qquad (m, m'\in M)$$
$$\leq \inf_{m,m'\in M}(p(x+m)+p(y+m'))$$
$$\leq \dot{p}(\dot{x})+\dot{p}(\dot{y}).$$

Thus $\dot{p}$ is subadditive.

Now, for all $\dot{x}\in E/M$ and all $\lambda\in C$,

$$\dot{p}(\lambda\dot{x})=\dot{p}(\lambda(x+M))$$
$$=\inf_{m\in M} p(\lambda(x+m))$$
$$=\inf_{m\in M}|\lambda|p(x+m)=|\lambda|\inf_{m\in M} p(x+m)=|\lambda|\dot{p}(\dot{x}).$$

Thus $\dot{p}$ is positively homogeneous of degree 1.

Hence $\dot{p}$ is a seminorm on E/M.

(ii) Let r be a continuous seminorm on E/M. To show that $\dot{\mathfrak{P}}$ is a basis of continuous seminorms on E/M, we need to show that there exist $\dot{q}\in\dot{\mathfrak{P}}$ and a constant $c>0$ such that for every $x+M\in E/M$,

$$r(x+M)\leq c\dot{q}(x+M)$$

Now let $x+M$ be a fixed element in E/M. We define a function $p: E\longrightarrow M$ by

$$p(x)=r(x+M).$$

Then we have, for $y, z\in E$,

$$p(y+z)=r(y+z+M)$$
$$\leq r(y+M)+r(Z+M)$$
$$=p(y)+p(z)$$
$$p(\alpha y)=r(\alpha y+M)$$
$$=|\alpha|r(y+M)$$
$$=|\alpha|p(y)$$

Thus p is a seminorm on E. Further more, since $p=r\cdot\phi$ is continuous on E, it follows that p is a continuous seminorm on E. Since $\mathfrak{P}$ is a basis of continuous seminorms on E, there exist a $q\in\mathfrak{P}$ and a constant $c>0$ such that for each $x\in E$, we have

$$p(x)\leq cq(x).$$

Now we observe that for every $y\in x+M$,

$$p(x)=p(y)\leq cq(y).$$

Hence $p(x)=\inf_{y\in x+M} p(y)\leq \inf_{y\in x+M} cq(y)=c\dot{q}(x)$.

This completes the proof.

Reference

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