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A Study on the Existence of Optimal Control

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1. Introduction

The existence of a time optimal control problem was first proved by A. F. Filippov [1] in 1962 and then it was generalized to Pontryagin problem by L. Cessari. The notion of attainable set was used by Emilio Roxin [2] to give a different proof for the existence of an optimal control. And this idea was used by L. W. Neustadt [3] to solve the same type of problem without convexity condition of the set $\tilde{Q}^+(t,x)$ for a linear control problem.

In any rate, all the proofs of the existence theorem based upon the fact that the subset $R(t,x)$—the subset in (t,x)-space in which all the admissible trajectories lie—is compact. However, for a practical problem, there is no reason why the $R(t,x)$ is always compact.

The purpose of this paper is to find a condition under which an optimal control exists for the problem when the set $R(t,x)$ is not compact.

2. Preliminaries

We shall consider the system

$$(2.1)\qquad \frac{dx^i}{dt}=f^i(t,x^1,\cdots,x^n,u^1,\cdots,u^m)\ \ (i=1,2,\cdots,n)$$

with $x^i(t_0)=x_0^i$. We introduce the vectors

$$x=(x^1,\cdots,x^n)$$
$$u=(u^1,\cdots,u^m)$$
$$f=(f^1,\cdots,f^n)$$

in Euclidean space with the usual norm $\|x\|^2=\Sigma(x^i)^2$. Then the system (2.1) can be written

$$(2.2)\qquad \frac{dx}{dt}=f(t,x,u):x(t_0)=x_0$$

The fundamental problem of control theory is of the following form:

Given (i) subsets $\mathcal{T}_0$ and $\mathcal{T}_1$ of E^{n+1},

(ii) functions $g_0:\mathcal{T}_0\longrightarrow E^1$ and $g_1:\mathcal{T}_1\longrightarrow E^1$,

(iii) a class of functions $\mathcal{A}$ such that, for each choice of $u\varepsilon\mathcal{A}$, the system (2.2) has a solution $\varphi(\ ;t_0,x_0,u)$ defined on $[t_0,t_1]$ satisfying the conditions

(a) $(t_0, \varphi(t_0)) \in \mathcal{T}_0$

(b) $(t_1, \varphi(t_1)) \in \mathcal{T}_1$

(c) $R^i(t, \varphi(t),\ u(t)) \geqslant 0,\ (i=1, 2, \cdots m)$

where $R^i : E^{n+m+1} \longrightarrow E^1$ is given.

We wish to find a control $u \in \mathcal{A}$ which minizes the functional

$$(2.3) \qquad J(u, t_0, x_0) = g_0(t_0, x_0) + g_1(t_1, x_1) + \int_{t_0}^{t_1} f^0(t, \varphi(t),\ u(t))dt$$

where $f^0 : E^{n+m+1} \longrightarrow E^1$ is a given function.

If R^i, $(i=1, 2, \cdots m)$, are sufficiently nice, then, for a fixed (t, x), condition (c) of (iii) gives a subset of E^m which we denote $\Omega(t, x)$. Then condition (c) of (iii) becomes $u(t) \in \Omega(t, \varphi(t))$ for each t in $[t_0, t_1]$. Thus the fundamental problem can be described as follows:

We wish to minimize the functional (2.3) subject to the restraints

(i) $\frac{dx}{dt} = f(t, x, u);\ x(t_0) = x_0$

(ii) $(t_0, x_0) \in \mathcal{T}_0$ *and* $(t_1, x_1) \in \mathcal{T}_1$

(iii) $u(t) \in \Omega(t, \varphi(t));\ t \in [t_0, t_1]$

where φ is a solution of (i) corresponding to the choice of a control $u \in \mathcal{A}$.

Throught this paper, R denotes a subset of the (t, x) -space E^{n+1} and $D = \{(t, x, u) \mid (t, x) \in R \text{ and } u \in \Omega(t, x)\}$.

Definition: A pair of functions (φ, u) defined on an interval $[t_0, t_1]$ is said to be an *admissible pair* if the following conditions hold:

(1) φ is absolutely continuous on $[t_0, t_1]$.

(2) u is measurable on $[t_0, t_1]$.

(3) $(t, \varphi(t)) \in R$ for every t in $[t_0, t_1]$.

(4) $u(t) \in \Omega(t, \varphi(t))$ for almost all t in $[t_0, t_1]$.

(5) $f^0(t, \varphi(t), u(t)) \in L_1[t_0, t_1]$.

(6) φ is a solution of the system

$$\frac{dx}{dt} = f(t, x, u(t)) \text{ with } \varphi(t_0) = x_0$$

(7) $(t_0,\ \varphi(t_0)) \in \mathcal{T}_0$ and $(t_1, \varphi(t_1)) \in \mathcal{T}_1$

Definition: We say that the class π of admissible pairs is *complete* if the following condition holds:

If $\{(\varphi_k, u_k)\}$ is a sequence of admissible pairs defined on $[t_{0k}, t_{1k}]$ and if $\varphi_k \longrightarrow \varphi$ in the Frechet sense, then there exists a measurable function u so that (φ, u) is admissible.

In this case, we shall call u an admissible control and φ the corresponding trajectory.

Definition: Let $\tilde{f} = (f^0, f^1, \cdots\cdots, f^n) = (f^0, f)$ and define

$$\tilde{Q}(t, x) = \{(z^0, z) \mid z^0 = f^0(t, x, u) \text{ and } z = f(t, x, u) \text{ with } u \in \Omega(t, x)\}$$

and $$\tilde{Q}^+(t, x) = \{(z^0, z) \mid z^0 \geqslant f^0(t, x, u) \text{ and } z = f(t, x, u) \quad \text{with } u \in \Omega(t, x)\}$$

Following theorem due to L. Cessari is well knowen.

Theorem 1. *Suppose following conditions are satisfied*

(1) *R is compact,*

(2) *Ω is an upper semicontinous function of R into 2^{E^m}, and for each (t, x) in R, $\Omega(t, x)$ is compact,*

(3) *$\tilde{f}$ is continuous on D,*

(4) *$\tilde{Q}^+$ (t, x) is a convex subset of E^{n+1} for every (t, x) in R, and*

(5) *$\mathcal{T}_0$ and $\mathcal{T}_1$ are closed subsets of E^{n+1} and g_0 and g_1 are continuous on $\mathcal{T}_0$ and $\mathcal{T}_1$, respectively.*

then the functional (2.3) *attains its minimum in any non-empty complete class of admissible pairs.*

Following example shows that if R is not compact, then the optimal control may not exist.

Example 1. For the system

$$\frac{dx}{dt}=2x^2(1-t)-1+u$$

with (1) $\Omega(t, x)=\{\|u\|\leqslant 1\}$

(2) $t_0=0$, $x_0\leqslant 1$, t_1 is fixed.

Consider the problem of maximizing $\varphi(t_1)$. For this problem, $f^0=0$ and thus $\tilde{Q}^+$ is convex. Take $u\equiv 1$. If we let $\varphi(0)=1$, then the corresponding solution is given by

$$\varphi(t)=\frac{1}{(1-t)^2+C}$$

where C is a constant real number. There are infinitely many solutions. Thus if $t_1\geqslant 1$, then there is no optimal solution. For this problem R is not compact.

3. Main Results

In this paper, we are going to study the possibility of replacing the condition R be compact by some other conditions.

Theorem 2. *Suppose that the conditions of Theorem* 1 *holds except condition* (1). *If*

(1) *R is closed and contained in a slab*

$$\{(t, x)\mid T_0\leqslant t\leqslant T_1 \text{ and } -\infty<x^i<\infty\}$$

(2) *there exists a constant $N>0$ such that, for all (t, x, u) in D, $\langle x, f\rangle\leqslant N(\|x\|^2+1)$ where $\langle\,,\,\rangle$ denotes the inner product.*

(3) *all trajectories have at least one point $(t, \varphi(t))$ belonging to a compact subset P of E^{n+1},*

then the conclusion of Theorem 1 *still holds.*

Proof: Let (φ, u) be an admissible pair and define

$$\Phi(t)=\|\varphi(t)\|^2+1.$$

Then

$$\frac{d\Phi}{dt}=2\sum_{i=1}^{n}\varphi^i(t)\frac{d\varphi^i}{dt}$$

$$= 2\sum_{i=1}^{n} \varphi^i(t) f^i(t, \varphi(t), u(t))$$
$$\leqslant 2N(\|\varphi\|^2+1)=2N\Phi(t)$$

Thus

$$\frac{d\Phi}{dt} \leqslant 2N\Phi(t).$$

Suppose $(t^*, (t^*))$ belongs to P. Integrating from t^* to t,

$$\Phi(t) \leqslant \Phi(t^*)e^{2N(t-t^*)} \leqslant \Phi(t^*)e^{2N(T_1-T_0)} \leqslant K$$

where K is a constant independent of φ. Hence φ lies in a compact subset of the (t, x)-space.

Theorem 3. *Suppose* $g_0=0$ *and* $g_1=0$ *and that the conditions of Theorem* 1 *holds except condition* (1). *If* R *is closed and, in addition to conditions* (2) *and* (3) *of Theorem* 2,

(4) *there exists* $G \geqslant 0$ *such that*

$$f^0(t, x, u) \geqslant -G \text{ for all } (t, x, u) \in D$$

(5) *there exists an* $N_1>0$ *and* $\mu>0$ *such that* $f^0(t, x, u) \geqslant \mu$ *for all* $(t, x, u) \in D$ *with* $|t|>N_1$ *then the conclusion of Theorem* 1 *still holds:*

Proof: Let $(\bar{\varphi}, \bar{u})$ be an admissible pair and suppose

$$J(\bar{\varphi}, \bar{u})=\int_{t_0}^{t_1} f^0(t, (\bar{\varphi}t), \bar{u}(t))dt=j$$

Let α be any positive number such that

(i) $\alpha \geqslant N$,

(ii) the projection of P on t-axis is contained in $[-\alpha, \alpha]$.

Let $\alpha_0=\alpha+L$ where $L>0$ which will be given later. We shall show that with an appropriate choice of L, if (φ, u) is an admissible pair with points $(t_3, \varphi(t_3))$ such that $|t_3|>\alpha_0$, then $J(\varphi, u) \geqslant j$. Thus for the minimizing problem we can ignore such trajectories and apply the result of Theorem 2.

Now there exists a point $(t^*, \varphi(t^*))$ in P which is on the trajectory. If $|t_3|>\alpha_0$ and $\varphi(t_3)$ is in the trajectory, then by the conditions (4) and (5) there exists an subinterval $[t', t'']$ of $[-\alpha, \alpha]$ on which

$$f^0(t, \varphi(t), u(t)) \geqslant -G$$

and for the remainder of the trajectory, call the corresponding interval E,

$$f^0(t, \varphi(t), u(t) \geqslant \mu>0$$

and thus we have

$$J(\varphi, u)=\int_{t'}^{t''} f^0(t, \varphi(t), \ u(t))dt+\int_E f^0(t, \varphi(t), u(t))dt \geqslant -2\alpha G+\mu L$$

Hence if we take

$$L=\mu^{-1}(2\alpha G+|j|+1)$$

then

$$J(\varphi, u) \geqslant |j|+1>j$$

4. Remarks

Condition (4) of Theorem 1 is necessary but by no means sufficient. That is, even if

$\tilde{Q}^+$ (t, x) is not convex, optimal control may still exists.

Example 2. Consider the problem of minimizing

$$J(u)=\int_0^1 (u^2-1)^2 dt$$

with

$$\frac{dx}{dt}=u, \quad \Omega(t,x)=\{\|u\|\leqslant 1\}$$

$$J_0=(0,0) \text{ and } J_1=(1,0).$$

For this problem,

$$f^0=(u^2-1)^2$$

so

$$(f^0)'=4(u^3-u)$$

$$(f^0)''=12u^2-4$$

Thus f^0 is not convex so that $\tilde{Q}^+$ is not convex. But the minimum of $J(u)$ still exists; that is, if we take u^*

$$u^*(t)=\begin{cases} 1 & \text{for} \quad 0\leqslant t\leqslant \frac{1}{2} \\ -1 & \text{for} \quad \frac{1}{2}\leqslant t\leqslant 1 \end{cases}$$

then $J(u^*)$ is the minimum.

In fact, for the linear control system, the convexity condition is not necessary; that is, if the control system is such that

$$f^0(t,x,u)=a(t)x+b^0(u,t)$$

$$f(t,x,u)=A(t)x+b(u,t)$$

where $x, b(u,t)$, and $a(t)$ are n-vectors, $b^0(u,t)$ is scalar, and $A(t)$ is an n-n matrix, then without the convexity condition of $\tilde{Q}^+$, an optimal solution exists [3].

References

1. A. F. Filippov, On certain questions in the theory of optimal control, *J. SIAM control,* Vol. 1, No. 1, 76-84 (1962)
2. Emilio Roxin, The existence of optimal controls, *Mich Math. J.* 9, 109-119 (1962)
3. L. W. Neustadt, The existence of optimal controls in the absence of convexity conditions, *J. Math. Anal. and Appl.* 7, 110-117 (1963)
4. E. B. Lee and L. Markus, *Foundations of optimal control theory,* The SIAM series in applied math, John Wiley (1967)
5. V. Jurdjevic, Abstract Control systems: Controllability and observability, *J. SIAM control,* Vol. 8, No. 3, 403-423 (1970)
6. Henning Tolle, *Optimization Methods,* Springer-Verlag (1975)
7. Andrew P. Sage, Chelsea C. White, 111, *Optimum Systems Control,* (second edition), Prentice-Hall (1977)