Optimal Selection of Populations for Units in a System

Woo-Chul Kim*

ABSTRACT

A problem of choosing units for the series system and the 1-out-of-2 system from k available brands is treated from a decision-theoretic points of view. It is assumed that units from each brand have exponentially distributed life lengths, and that the loss functions are inversely proportional to the reliability of the system. For the series system the 'natural' rule is shown to be optimal. For the 1-out-of-2 system, the Bayes rule wrt the natural conjugate prior is derived and the constants to implement the Bayes rule are given.

1. Introduction

Suppose we have an l-out-of-m system, where m units are to be placed and at least l of them should function to make the system work, and the life lengths of the units are independent. In many situations we have k available populations (brands) $\Pi_1, \ldots, \Pi_k$, and we need to decide which population we are going to use for each unit.

Assume that each unit from the i-th population has an exponentially distributed life length with mean life length $\lambda_i^{-1} (i=1, \ldots, k)$. Now the problem is to find an optimal selection rule based on n independent observations $X_{i_1}, \ldots, X_{i_n}$ from each Π_i $(i=1, \ldots, k)$. By sufficiency the problem

* Department of Computer Science and Statistics, Seoul National University. This work was supported in part by the Research Fund of the Ministry Education, Korean Government, in 1997.

can be reduced to the one based on $X_1, \ldots, X_k$ where $X_i=\sum_{j=1}^{n} X_{i_j}$ has a gamma distribution with mean $n\lambda_i^{-1}$ and variance $n\lambda_i^{-1}$.

In section 2, we consider a series system, i.e., m-out-of-m system. The reliability of the system, then, is easily seen to be $(\sum_{j=1}^{m} \lambda_{i_j})^{-1}$ when the populations $\Pi_{i_1}, \ldots, \Pi_{i_m}$ $(1\leq i_1\leq\cdots\leq i_m\leq k)$ are chosen (see, for example, [2]). Hence it seems reasonable to take $\sum_{j=1}^{m} \lambda_{i_j}$ as the loss incurred by such a selection since it is inversely proportional to the reliability. Then it is shown that the natural rule, which draws all the units from the population associated with $\max_{1\leq i\leq k} X_i$, is uniformly best among the permutation invariant rules and, therefore, it is admissible and minimax.

In sections 3, the Bayes rule for the 1-out-of-2 system is derived where the loss incurred by selecting Π_i and Π_j $(1\leq i\leq j\leq k)$ is assumed to be the reciprocal of the reliability and the natural conjugate prior is assumed. The table of the constants to implement the Bayes rule is also given.

Throughout the paper let $x_{(1)}\leq\cdots\leq x_{(k)}$ denote the ordered observations of $X_1,\cdots, X_k$, and $\Pi_{(i)}$ and $\lambda_{(i)}$ denote the Π and λ associated with $x_{(i)}$ for $i=1,\cdots, k$. Given $\underline{X}=\underline{x}=(x_1,\cdots, x_k)$, the posterior risk of a decision rule d will be denoted by $r(d,\underline{x})$.

2. The Series System

Here, we can take the action space by $A=\{(i_1,\cdots i_m) : 1\leq i_1\leq\cdots\leq i_m\leq k\}$ where $(i_1,\cdots, i_m) \varepsilon A$ is to be interpreted as the action of drawing the j-th unit from Π_{i_j} for $j=1, \cdots, m$. The loss function is assumed to be

$$L(\lambda, (i_1,\cdots, i_m))=\sum_{j=1}^{m} \lambda_{i_j} \tag{2.1}$$

which is the reciprocal of the reliability of the system. Hence the posterior risk of a dicision rule d, which takes an action $(i_1,\cdots i_m) \varepsilon A$ with probability 1 is given by

$$r(d,\underline{x})=E\left[\sum_{j=1}^{m}\lambda_{i_j}|\underline{x}\right]. \tag{2.2}$$

Let $N_s=\{\underline{n}=(n_1,\cdots,n_s): n_1\geq\cdots\geq n_s,\ \sum_{j=1}^{s}n_j=m,\ n_j\in Z,\ j=1,\cdots,s\}$ where Z is the set of positive integers, and let $a_\wedge b=\text{Min}(a,b)$. The next result leads to considerable reduction in the number of decision rules to be compared when the prior of $\underline{\lambda}$ is assumed to be permutationally symmetric.

Lemma 2.1. Assume that the prior distribution of $\underline{\lambda}$ is permutationally symmetric. Then the Bayes rule d^* is determined by

$$r(d^*,x)=\underset{1\leq s\leq k\wedge m}{\text{Min}}\ \underset{n\in Ns}{\text{Min}}\ r(d_{\underline{n}},\underline{x}) \tag{2.3}$$

where $d_{\underline{n}}$ selects an action of drawing n_j units from $\Pi_{(k-j+1)}$ for $j=1,\ldots,s$.

Proof. The action space A can be partitioned into $k_\wedge m$ components $A_s(s=1,\ldots,k_\wedge m)$ where we choose s different populations for m units. Note that A_s can be written as $A_s=\underset{\underline{n}\in Ns}{U}A_{s,n}$, where $A_{s,\underline{n}}$ is defined as follows: for given $s=1,\ldots,k_\wedge m$, and $\underline{n}\varepsilon N_s$, let

$$A_{s,\underline{n}}=\{(i_1,\ldots,i_s;n_1,\ldots,n_s):\ 1\leq i_j\leq k,\ i_j\neq i_{j'}\ \text{for}\ j\neq j'\}$$

where $(i_1,\ldots,i_s;n_1,\ldots,n_s)$ is a shorthand for the action of drawing $n_1,\ldots,n_s$ unist from $\Pi_{i_1},\ldots,\Pi_{i_k}$.

Now consider a decision problem in which the action space is given by $A_{s,\underline{n}}$ and the loss is given by (2.1), i.e., $L(\underline{\lambda},a)=\sum_{j=1}^{s}n_j\lambda_{i_j}$ for $a\varepsilon A_{s,n}$. Clearly this problem is equivalent to partitioning k populations $\Pi_1,\ldots,\Pi_k$ into $s+1$ subsets $(\gamma_1,\ldots,\gamma_s,\gamma_{s+1})$ where γ_j is of size 1 for $j=1,\ldots,s$, γ_{s+1} is of size k-s. Note that this decision problem is invariant under the permutation group, and that the loss function satisfies the monotonicity and the invariance of Eaton [4]. Since the density $f(x,\lambda_i)$ of X_i has the monotone likelihood ratio (MLR) in x and λ_i^{-1}, it follows from Eaton's result that the rule which assigns $\Pi_{(k-j+1)}$ to γ_j for $j=1,\ldots,s$ and the remaining ones to γ_{s+1} is Bayes wrt any permutationally symmetric prior. Hence the result follows.

The following lemma is needed for the main result.

Lemma 2. 2. Assume that $X_1,\dots, X_k$, given $(\theta_1, \dots, \theta_k) \varepsilon Ⓗ^k$, are independently distributed random variables with X_i having pdf $f(x,\theta_i)$. If $f(x,\theta)$ has the MLR property in x and θ, and if the prior distribution, $\tau(\underline{\theta})$, of $\underline{\theta}=(\theta_1,\dots,\theta_k)$ is permutationally symmetric on Ⓗ^{k}, then, for $i\geq j$,

$$E[g(\theta_{(i)})\,|\,\underline{x}]\geq E[g(\theta_{(j)})\,|\,\underline{x}]$$

provided $g(\cdot)$ is non-decreasing on Ⓗ and $\theta_{(i)}$ is the θ associated with $x_{(i)}$.

Proof. Let $\Omega_0=\{\underline{\theta}\varepsilon Ⓗ^k : \theta_{(i)}\geq\theta_{(j)}\}$, then

$$\int_{\Omega_0 Ⓗ^k}[g(\theta_{(i)})-g(\theta_{(j)})]\ f(\underline{x},\underline{\theta})\ d\tau(\theta)$$
$$=[(\int_{\Omega_0^c}+\int_{\Omega_0}]\ [g(\theta_{(i)})-g(\theta_{(j)})]\ f(\underline{x},\underline{\theta})\ d\tau(\underline{\theta})$$
$$=\int_{\Omega_0}[g(\theta_{(i)})-g(\theta_{(j)})]\ [f(\underline{x},\underline{\theta})-f(\underline{x},\underline{\theta})]d\tau(\underline{\theta})$$

where $f(\underline{x},\underline{\theta})=\prod_{i=1}^{k} f(x_i,\theta_i)$ and $\underline{\theta}'$ is obtained from $\underline{\theta}$ by interchanging $\theta_{(i)}$ and $\theta_{(j)}$, keeping other components fixed. The result follows from the MLR property of $f(\underline{x}, \underline{\theta})$ and the fact that $g(\theta_{(i)})-g(\theta_{(j)})\geq 0$ for $\theta\ \varepsilon\ \Omega_0$.

Remark 2. 1. The MLR property and the independence can be replaced by the DT (decreasing in transposition) property of $f(\underline{x}, \underline{\theta})$ in Hollander, Proschan and Sethuraman [6]. Now we state the following result.

Theorem 2. 1. For any permutationally symmetric prior of $\underline{\lambda}$, the Bayes rule d^* draws all m units from $\Pi_{(k)}$.

Proof. It follows from the definition of $d_{\underline{n}}$ and (2. 2) that, for $\underline{n}\ \varepsilon N_s$.

$$r(d_{\underline{n}}, \underline{x})=E\left[\sum_{j=1}^{s} n_j\lambda_{(k-j+1)}\,|\,\underline{x}\right]$$

Therefore, for $\underline{n}\ \varepsilon N_s$,

$$r(d_{\underline{n}}, \underline{x})-E\left[(m-s+1)\lambda_{(k)}+\sum_{j=2}^{s}\lambda_{(k-j+1)}\,|\,\underline{x}\right]$$
$$=E\left[(m-\sum_{j=2}^{s} n_j)\lambda_{(k)}+\sum_{j=2}^{s} n_j\lambda_{(k-j+1)}\,|\,\underline{x}\right]$$
$$=E\left[(m-s+1)\lambda_{(k)}+\sum_{j=2}^{s}\lambda_{(k-j+1)}\,|\,\underline{x}\right]$$
$$=E\left[\sum_{j=2}^{s}(n_j-1)(\lambda_{(k-j+1)}-\lambda_{(k)})\,|\,\underline{x}\right]$$
$$\geq 0,$$

by Lemma 2. 2.

Thus $\underset{n \epsilon N_s}{\text{Min}}\ r(d_{\underline{n}}, \underline{x})=r(d_s,\underline{x})$ where d_s draws$(m-s+1)$ units from $\Pi_{(k)}$ and one unit from each $\Pi_{(k-j+1)}(i=2,\ldots, s)$. On the other hand, for any $s: 2\leq s\leq k_\wedge m$

$$r(d_s, \underline{x})-r(d_1,\underline{x})=E\left[\sum_{j=2}^{s}(\lambda_{(k-j+1)}-\lambda_{(k)})\right]$$
$$\geq 0,$$

again by Lemma 2. 2. Hence $d^*=d_1$, i.e., the Bayes rule draws all m units from $\Pi_{(k)}$.

The next result follows from considering a permutationally symmetric prior which gives mass $1/k!$ at each parameter point obtained from any given parameter $\underline{\lambda}$ by permuting its components. (see, for example, §4. 3 in Ferguson [5])

Corollary 3. 1. The natural rule d^* is uniformly best among the permutation invariant rules, and it is admissible and minimax among all dicision rules.

Remark 2. 2. If we consider a loss function $L_1(\underline{\lambda}(i_1,\ldots, i_m))=(m \underset{1\leq i\leq k}{\text{Min}} \lambda_i)^{-1} -\left(\sum_{j=1}^{m} \lambda_{i_j}\right)^{-1}$, it can be easily shown that Lemma 2. 1. holds for this loss function. Assuming an exchangeable prior for $\underline{\lambda}$, it can be verified that the Bayes rule d^* is determined by $r(d^*, \underline{x}) =\underset{1\leq s\leq k_\wedge m}{\text{Min}} r(d_s, \underline{x})$ where the rule d_s is as in the proof of Theorem 2. 1. Even though this is a considerable reduction in a number of candidates for the Bayes rule, specification of it seems difficult except when $m=2$. Further simplification of the Bayes rule with respect to a specific prior would be interesting along with some numerical results.

3. The 1-out-of-2 System

Here, the action space is $A=\{(i, j): 1\leq i\leq j\leq k\}$ where $(i, j)\, \varepsilon A$ denotes the action of drawing one unit each from Π_i and Π_j. For the 1-out-of-2 system, the reliability of the system corresponding to the action $(i, j) \varepsilon A$ is given by $\lambda_i^{-1}+\lambda^{-1}{}_j-(\lambda_i+\lambda_j)^{-1}$. Hence it seems reasonable to assume that the loss function is given by

$$L(\underline{\lambda}, (i, j))=(\lambda_i^{-1}+\lambda_j^{-1}-(\lambda_i+\lambda_j)^{-1})^{-1} \tag{3.1}$$

Broström [3] considered the 1-out-of-2 system when only two populations are available, and assumed a loss function depending on (λ_1, λ_2) only through λ_1/λ_2 so that the problem is invariant under the scale transformation. This could be achieved by dividing (3.1) by $L(\lambda,(1,2))$, i.e., the loss incurred by an "intermediate" action, which does not exist in the case when $k>2$.

We will consider a natural conjugate prior τ of $\underline{\lambda}$, given by the pdf

$$\tau(\underline{\lambda})=\prod_{i=1}^{k}\left[\frac{\beta^{\alpha}}{\Gamma(\alpha)}\ \lambda_i^{\alpha-1}\ e^{-\beta\lambda_i}\right],\ \alpha<0 \text{ and } \beta>0 \tag{3.2}$$

i.e., $\lambda_1,\dots, \lambda_k$ are, a priori, independent gamma random variable with common scale and shape parameters, β^{-1} and α, respectively. It can be easily observed that, given $\underline{Y}=\underline{x}$, the $\lambda_{(i)}$ are, a posteriori, independently distributed gamma random variables with mean $(n+\alpha)(x_{(i)}+\beta)^{-1}$ and variance $(n+\alpha)(x_{(i)}+\beta)^{-2}$.

The action space A can be partitioned into $A_1=\{(i,i) : i=1,\dots, k\}$ and $A_2=\{(i,j) : 1\leq i<j\leq k\}$. Then the decision problem with the action space A_s $(s=1,2)$ and the loss function in (3.1) is equivalent to partitioning $\Pi_1,\dots, \Pi_k$ into two subsets γ_1 and γ_2 with γ_1 being of size s and γ_2 being of size $k-s$. Hence, by the same arguments as in Lemma 2.1, we have the next result.

Lemma 3.1. Assume that the prior of $\underline{\lambda}$ is permutationally symmetric. Then the Bayes rule d^* is given by,

$$r(d^*, \underline{x})=\text{Min}\ \{r(d_1, \underline{x}),\ r(d_2, \underline{x})\} \tag{3.3}$$

where d_1 chooses 2 units from $\Pi_{(k)}$ and d_2 chooses 1 unit from $\Pi_{(k)}$ and another from $\Pi_{(k-1)}$.

Theorem 3.2. Assume that the prior is given by (3.2). Then the Bayes rule d^* is given by

$$d^*=\begin{cases} d_1 & \text{if } x_{(k-1)}+\beta\leq c(x_{(k)}+\beta) \\ d_2 & \text{if } x_{(k-1)}+\beta> c(x_{(k)}+\alpha) \end{cases} \tag{3.4}$$

where $c\varepsilon$ (0, 1) is determined by $H_{\alpha,n}(c)=E\left[\frac{UV(U+cV)}{U^2+c^2V^2+cUV}\right]=\frac{2}{3}(n+\alpha)$ and U, V are iid gamma random variables with mean $n+\alpha$ and variance $n+\alpha$.

Proof. It follows from (3.1) and (3.2) that

$$r(d_1, \underline{x})=\frac{2}{3}\ E[\lambda_{(k)}\,|\,\underline{X}]=\frac{2}{3}\cdot\frac{n+\alpha}{x_{(k)}+\beta} \text{ and}$$

$$r(d_2,\underline{x})=E[\lambda_{(k)}\lambda_{(k-1)}(\lambda_{(k)}+\lambda_{(k-1)})(\lambda_{(k)}{}^2+\lambda_{(k)}\lambda_{(k-1)}+\lambda_{(k-1)}{}^2)^{-1}\,|\,x]$$

$$=\frac{1}{x_{(k)}+\beta}\ E\left[\frac{UV(U+rV)}{U^2+rUV+r^2V^2}\right],$$

where $r=(x_{(k-1)}+\beta)/(x_{(k)}+\beta)$, U and V are iid gamma random variables with mean $(n+\alpha)$ and variance $(n+\alpha)$. Since $H_{\alpha,n}(t)$is non-decreasing in $t>0$, $r(d_1, \underline{x})\geq r(d_2,\underline{x})$ if and only if $(x_{(k-1)}+\beta)/(x_{(k)}+\beta)\geq c$. Furthermore, it can be easily shown that $H_{\alpha,n}(1)<\frac{2}{3}(n+\alpha)$, which implies $0<c<1$. Hence the result follows from Lemma 3.2.

It can be easily shown that $X_1(X_1+\beta)^{-1}, \ldots, X_k(X_k+\beta)^{-1}$ are marginally independent beta random variables with mean $n(n+\beta)^{-1}$. Hence the overall risk of the rule d_1 is given by

$$r(d_1)=\frac{2}{3}\ (n+\alpha)\beta^{-1}\ E[Z_{(1)}]\leq\frac{2}{3}\ \frac{\alpha}{\beta}$$

where $Z_{(1)}$ is the smallest order statistic based on a sample of size k from beta distribution with mean $\alpha(n+\alpha)^{-1}$. Therefore, the overall risk of the Bayes rule d^* is finite. Furthermore, it can be shown that the risk function $R(\underline{\lambda}, d)$ is a continuous function of $\underline{\lambda}$ for any rule d(see, for example, § 3.7 in Ferguson [5]). Hence, we have the next result.

Corollary 3.1. The Bayes rule given by (3.4) is admissible.

Note that the generalized Bayes rule wrt the vague prior is given by

$$d^G=\begin{cases} d_1 & \text{if } x_{(k-1)}\leq bx_{(k)} \\ d_2 & \text{if } x_{(k-1)}> bx_{(k)} \end{cases}$$

where $b\varepsilon(0,1)$ is given by $H_n(\mathrm{b})=E\left[\frac{UV(U+bV)}{U^2+b^2V^2+bUV}\right]=\frac{2}{3}n$, i.e., the generalized Bayes rule uses 2 units from $\Pi_{(k)}$ only when the largest sample mean life length and the second largest sample mean life length are really different.

Remark 3.1, Suppose that the loss incurred by (i,j). εA is given by $\frac{3}{2}(\operatorname*{Min}_{1\leq i\leq k}\lambda_i)^{-1}-[\lambda_i^{-1}+\lambda_j^{-1}-(\lambda_i+\lambda_j)^{-1}]$. Then it follows in a similar way that the Bayes rule wrt the prior given by (3.2) is of the same form in (3.4) except that $c\ \varepsilon\ (0,1)$ is determined by

$$G_{\alpha,n}(c)=\frac{1}{2}\ \frac{1}{n+\alpha-1}-\frac{c}{n+\alpha-1}+E\left[\frac{c}{cU+V}\right]=0$$

where U and V are independent gamma random variables with mean and variance equal to $(n+\alpha)$.

The constants to implement the (generalized) Bayes rules given in Theorem 3.1. and Remark 3.1. are found by numerically integrating $H_{\alpha,n}(c)$ and $G_{\alpha,n}(c)$ using the first fifteen Laguerre polynomials (see Abramowitz and Stegun [1]). These are given at the end of the paper.

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Table. 1. Lists c-values to implement the (generalized) Bayes rule in Theorem 3.1. which depends on n and α through the quantity $m=n+\alpha$.

m	c	m	c	m	c
1.0	.3033	11.0	.8870	21.0	.9388
1.5	.4392	11.5	.8916	21.5	.9402
2.0	.5342	12.0	.8958	22.0	.9415
2.5	.6021	12.5	.8997	22.5	.9428
3.0	.6530	13.0	.9034	23.0	.9440
3.5	.6925	13.5	.9067	23.5	.9451
4.0	.7240	14.0	.9099	24.0	.9462
4.5	.7496	14.5	.9128	24.5	.9473
5.0	.7710	15.0	.9156	25.0	.9483
5.5	.7890	15.5	.9182	25.5	.9493
6.0	.8044	16.0	.9206	26.0	.9502
6.5	.8177	16.5	.9229	26.5	.9511
7.0	.8293	17.0	.9251	27.0	.9520
7.5	.8396	17.5	.9271	27.5	.9529
8.0	.8486	18.0	.9291	28.0	.9537
8.5	.8567	18.5	.9309	28.5	.9545
9.0	.8640	19.0	.9326	29.0	.9552
9.5	.8706	19.5	.9343	29.5	.9560
10.0	.8766	20.0	.9359	30.0	.9567
10.5	.8820	20.5	.9374	30.5	.9574

Table 11. Lists c-values to implement the (generalized) Bayes rule in Remark 3.1. which depends on n and α through the quantity $m=n+\alpha$.

m	c	m	c	m	c
1.5	.7854	11.5	.9817	21.5	.9904
2.0	.8592	12.0	.9825	22.0	.9907
2.5	.8957	12.5	.9832	22.5	.9909
3.0	.9173	13.0	.9839	23.0	.9911
3.5	.9315	13.5	.9845	23.5	.9913
4.0	.9415	14.0	.9851	24.0	.9915
4.5	.9490	14.5	.9856	24.5	.9916
5.0	.9548	15.0	.9861	25.0	.9918

m	c	m	c	m	c
5.5	.9594	15.5	.9866	25.5	.9920
6.0	.9631	16.0	.9870	26.0	.9921
6.5	.9662	16.5	.9874	26.5	.9923
7.0	.9688	17.0	.9878	27.0	.9924
7.5	.9711	17.5	.9882	27.5	.9926
8.0	.9730	18.0	.9885	28.0	.9927
8.5	.9747	18.5	.9888	28.5	.9928
9.0	.9762	19.0	.9891	29.0	.9930
9.5	.9776	19.5	.9894	29.5	.9931
10.0	.9788	20.0	.9897	30.0	.9932
10.5	.9798	20.5	.9900	30.5	.9933
11.0	.9808	21.0	.9902	31.0	.9934