# Control and Aggregation (II)

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In the last paper[1], we have discussed the cannonical representation of a dynamic linear model, on which some aggregation schemes were devised. The relationships of those aggregation schemes with dynamic properties were investigated. This paper tries to analyse she control strategy for the aggregated linear dynamic model and to investigate the dynamic properties of disaggregative model controlled by aggregated model.

For the logical consistency with the last paper, all the sections and all the equations are numbered in a sequence.

## 5. Control of the Aggregative Model

Before discussing the main arguments, a rationale on the control by the aggregation of the original model is discussed. We have discussed what the purpose of the control should be in Chapter II. First of all, the purpose of long-run policy is the stabilization of the economic system.

The convergence of the Ricatti difference equation has been shown to be a necessary and sufficient condition for stabilization policy. The coefficient matrix of the lagged endogenous variables and instruments in the linear feedback equation is only a function of the steady-state solution of the Ricatti difference equation. This is a function of the coefficient matrix of the model and the weight matrix of the loss function, but not of the desired target levels or uncontrollable exogenous variables. Secondly, the purpose of

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[1] Sung-Shin Han, "Control and Aggregation(I)", *The Journal of The Korean Statistical Society*, December, 1979

control is for the attainment of the target levels for short-run policy. This is determined by the constant terms of the linear feedback equation. In the spirit of optimization, this term is also a function of iterative matrix solutions of the Ricatti difference equation and target levels and the uncontrollable exogenous variables through the time horizon. But when we consider uncertainty of target levels for the long-run and formidable prediction errors of uncontrollable exogenous variables for long-run forecasting, the long-run policy based on the target levels may be meaniningless. If the coefficient matrix of the econometric model is constant over time, the coefficient matrix of the lagged endogenous variables in the linear feedback equation should be constant when either she moving time horizon or the infinite time horizon is used. But the intercept of the linear feedback equation is not constant anymore whatever time horizon is used. This is because of the changing target level, uncontrollable exogenous variables, prediction error on those variables, and, moreover, fine tuning or constant adjustment of the model in each point of time. Thus recomputation of the intercept term would be unavoidable, unlike the coefficient matrix of the lagged endogenous variables in the linear feedback equation. This coefficient matrix should be represented by one policy rule inflexible to the target level and another one which depends upon the prediction ability of the uncontrollable exogenous variables. We should note that an inflexible policy rule does not mean Friedman's constant growsh rate of the money supply. Suppose Friedman's constant growth is represented by $x_t = Gx_{t-1}$. When this rule is applied to model (1. 1), the derived reduced reduced form will be

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} A & -BG \\ 0 & G \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix}$$

Looking into this model, it is clear that if $y_t = Ay_{t-1}$ is unstable, the controlled system is also unstable because

$$\det\left( \begin{pmatrix} A & -BG \\ O & G \end{pmatrix} - \lambda I \right) = \det\ (A - \lambda I)\ \det\ (G - \lambda I)$$

where det (•) denotes the determinant.

Now returning to the rationale of aggregation for control, first, it can be shown that the loss function on the infinite time horizon can be represented as

$$(5.1) \qquad W=\sum_{t=1}^{\infty} y'_{t-1}\, Q\, y_{t-1}+x_t'\, R\, x_t$$

$$= \left(\sum_{t=1}^{T} {y_t}'_{-1}\, Q\, y_{t-1}+x_t'\, R\, x_t\right) +y_T' K_T y_T.$$

We are discarding target levels from the loss function without loss of generality because target levels are irrelevant to stabilization policy, if there exists the infinite time control rule. Looking at loss function (5. 1) it is easy to see that it is nothing but a finite-time horizon control problem with an appropriate of the terminal condition, $y_T' K_T y_T$. Furthermore, Chapter II shows that there exists a time-horizon $T$ such that the controlled system has characteristic roots within the unit circle. This is when the feedback equation is computed with $K_t$ where t is less than $T$ if and only if there exists the infinte-time horizon control rule. $K_t$ also has been shown to be the positive-definite and monotonic sequence with the finite elements for all time $t$ such that $t$ belongs to all positive integers. Though such $K_t$'s are proved to be unique, there may exist an approximate computation of $K_t$ such that the controlled system has characteristic roots within the unit circle. Suppose that the linear feedback equation is computed from the approximate $K_t$or $\tilde{K}_t$ as

$$(5.2) \qquad x_t=-\tilde{G}\, y_{t-1}+\tilde{g}_t$$

Then we can reformulate the control problem as

$$(5.3) \qquad \underset{\tilde{g}_t}{\text{Min}}\ E\ W=\underset{\tilde{g}_t}{\text{Min}}\ E\{\sum_{t=1}^{T} \{(y_{t-1}-y^*_{t-1})'Q(y_{t-1}-y^*_{t-1})$$

$$+(x_t-x_t^*)'R(x_t-x_t^*)\}+(y_T-y_T^*)'\ \tilde{K}_T(y_T-y_T^*)\}$$

subject to

$$(5.4) \qquad \underset{n\times 1}{y_t}=A y_{t-1}+Bx_t+ f_t+u_t$$

$$\underset{m\times 1}{x_t}=-\tilde{G}\,y_{t-1}+\tilde{g}_t$$

where'*' denotes the target level, $f_t$ denotes an uncontrollable exogenous variable vector, '$E$' denotes expected value and $u_t$ is a white noise. With the above formulation, the instrument vector is $\tilde{g}_t$ rather than $x_t$. Though the error term, $u_t$ exists, continuous application of Theil's certainty equivalence principle in the open-loop fashion would lead to an optimum $g_t$ in the formulations (5. 3) and (5. 4). Though the modified model (5. 4) with the dimension $(n+m)$ is fairly large enough, there are a lot of computational algorithms as quadratic programming problem on the large-scale model which have been studied by many economists (16). Thus the main purpose of the aggregation of the model to reduce the dimension is to obtain an estimate of $\tilde{G}$, not $\tilde{g}_t$. But in general, the existence of stabilization policy on the aggregative model does not imply such on the disaggregative model. To see this point clearly suppose that there exists the stabilization policy on the aggregative model, so

(5. 5) $$|\lambda(F-DC_M)|<1$$

where $|\lambda(\cdot)|$ denotes the absolnte value of the characteristic roots,

(5. 6) $$G_M=(R+D'K_MD)^{-1}D'K_MF,$$

(5. 7) $$K_M=Q_M+F'K_M(I+DR^{-1}D'K_M)^{-1}F, \text{ and}$$

(5. 8) $$W_M=\sum_{t=1}^{\infty}(z'_{t-1}Q_MZ_{t-1}+x'_tRx_t).^{2)}$$

The feedback equation can be derived from equation (5. 6) as

(5. 9) $$\begin{aligned}x_t&=-G_MZ_{t-1}\\&=-G_MCy_{t-1}\end{aligned}$$

Substuting equation (5. 9) into equation (1. 1)

(5. 10) $$y_t=(A-B\,G_MC)\,y_{t-1}.$$

From equation (5. 6), equation (2. 4), and equation (2. 5),

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2) $Q_M$ Should be constructed such that $C'Q_MC$ could be $Q$ as close as possible. Thus the least square solution would be

$$Q_M=(CC')^{-1}\;CQC'(CC')^{-1}$$

$$(5.11)\qquad \underset{n\times m}{H}=A-B\,G_MC=A-B(R+D'K_MD)^{-1}D'K_MFC$$

$$=A-B(R+B'\tilde{K}B)^{-1}B'\tilde{K}A$$

$$=(I+BR^{-1}B'\tilde{K})^{-1}A$$

where

$$(5.12)\qquad \underset{n\times n}{\tilde{K}}=\underset{n\times l}{C'}\quad\underset{l\times l}{K_M}\quad\underset{l\times n}{C}$$

To see the relationship between equations (5. 11) and (5. 5), we can consider a similar transformation of matrix $H$,

$$(5.13)\qquad H=\binom{C}{C_1}H\binom{C}{C_1}^{-1}$$

where the rectangular matrix $C_1$ is an arbitrary $(n-l)\times n$ matrix such that the $n\times n$ square matrix, $\binom{C}{C_1}$ is non-singular.

Then, from equation (5. 12),

$$(5.14)\qquad \tilde{H}=\{\binom{C}{C_1}(I+B\ R^{-1}\ B'\tilde{K})^{-1}\}\ \binom{C}{C_1}^{-1}\binom{C}{C_1}\ A\ \binom{C}{C_1}^{-1}$$

$$=\{I+\binom{C}{C_1}B\ R^{-1}\ B'\tilde{K}\ \binom{C}{C_1}^{-1}\}^{-1}\ \binom{C}{C_1}\ A\ \binom{C}{C_1}^{-1}$$

Let

$$(5.15)\qquad \binom{C}{C_1}^{-1}=(C^-C^-{}_1),$$

and then the following matrix identities hold,

$$(5.16)\qquad CC^-=\underset{l\times l}{I}$$

$$CC^-{}_1=\underset{l\times(n-l)}{0}$$

$$C_1C^-=\underset{(n-l)\times l}{0}$$

$$C_1C_1{}^-=\underset{(n-l)\times(n-l)}{I}$$

Equation (5. 14) can be rewritten as

$$(5.17)\qquad \tilde{H}=\left[I+\binom{C}{C_1}B\,R^{-1}\,B'C'K_M\,C(C^-C^-{}_1)\right]^{-1}\binom{C}{C_1}\ A\ (C^-C^-{}_1)$$

$$=\begin{bmatrix}I+C\ B\ R^{-1}\ B'C'K_M & 0\\ C_1\ B\ R^{-1}\ B'C'K_M & 1\end{bmatrix}^{-1}\begin{pmatrix}CAC^- & CAC_1{}^-\\ C_1AC^- & C_1AC_1{}^-\end{pmatrix}$$

$$=\begin{bmatrix}(I+CBR^{-1}B'C'K_M)^{-1}CAC^- & (I+CBR^{-1}B'C'K_M)^{-1}CAC_1^- \\ -C_1BR^{-1}B'C'K_MCAC^-+C_1AC^- & -CBR^{-1}B'C'K_MCAC_1^-+C_1AC_1^-\end{bmatrix}$$

From equation (2.4), (2.5), and (5.16),

$$\tilde{H}=\begin{bmatrix}(I+DR^{-1}D'K_M)^{-1}\bar{H} & 0 \\ -C_1BR^{-1}D'K_MF+C_1AC^- & C_1AC_1^-\end{bmatrix} \tag{5.18}$$

The characteristic roots of $\tilde{H}$ consist of the characteristic roots of $(I+DR^{-1}D'K_M)^{-1}F=(F-DG_M)$ and $C_1AC_1^-$. In order for the characteristic roots of $\tilde{H}$ to be within the unit circle, the characteristic roots of $C_1AC_1^-$ should be within the unit circle because the characteristic roots of $(F-DG_M)$ have been asssumed to be within the unit circle in equation (5.5), Now let us determine what is the sufficient condition for $C_1AC_1^-$ to have characteristic roots in the unit circle. Using the first form of the canonical representation (1.2),

$$C_1AC_1^-=C_1P\varLambda P^{-1}C_1^- \tag{5.19}$$

and

$$\varLambda=\begin{bmatrix}\gamma_1 & & & & & \\ & \lambda_2. & & & & \\ & & \cdot\ \lambda_l & & & \\ & & & \lambda_{l+1}. & & \\ & 0 & & & \lambda_n\end{bmatrix}$$

assuming that $|\lambda_1|>|\lambda_2|>\ldots>|\lambda_l|>|\lambda_{l+1}|>\cdots>|\lambda_n|$ and $|\lambda_{l+1}|<1$

If $C_1$ and $C_1^-$ are constructed by

$$C_1=\begin{bmatrix}q \\ q_n\end{bmatrix},\ C_1^-=\{p,\ldots,p_n\} \tag{5.20}$$

where $q_i$ and $p_i$ are defined as in Section 1, then clearly $C_1AC_1^-$ would have $\lambda_l,\ldots,\lambda_n$ as characteristic roots, which is in the unit circle. This is nothing but the first example of aggregation in Section. 3. Therefore, if the aggregative model is constructed in view of the characteristic vectors corresponding to the largest characteristic roots, there would exist stabilization policy on the aggregative model implying stabilization policy on the disaggregative model. As noted previously, this is not realistic in fact, because, in the case of the large-scale model, the computation of the characteristic vectors and roots might be mcre difficult than control of the disaggregative model. We have discussed an approximate solution to this problem in Section 4. We also mentioned the importance of the interaction between the coefficient matrix of the lagged endogenous variables, A, and the coefficient matrix of the instrument, B, because the multiplier matrix has the form of multiplication of $A^t$ and $B$. So the second form of the canonical representation is used. Thus, at first, the second form of the canonical representation is assumed along with the nearly completely decomposable or weakly coupled system. Thus, without loss of generality, the following model is assumed,

$$(5.21)\qquad \begin{array}{c} l \\ n-l \end{array}\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} + \begin{array}{c} l \\ n-l \end{array}\overset{m}{\begin{pmatrix} I \\ 0 \\ \cdots \\ 0 \end{pmatrix}} x_t$$

where $l \geq m$.

Suppose model (5.21) is weakly coupled or $\tilde{A}_{12}$ and $\tilde{A}_{21}$ are sufficiently small and the following conditions are satisfied,

$$(5.22)\qquad |\lambda_{\min}(\tilde{A}_{11})| > |\lambda_{\max}(\tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12})|$$

and

$$(5.23)\qquad |\lambda_{\max}(\tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{11}^{-1}\tilde{A}_{12})| < 1$$

where '<' denotes 'smaller than'. Then the aggregative model to control can be considered as

$$(5.24)\qquad Z_{1t} = \tilde{A}_{11} Z_{1t-1} + \binom{I}{0} x_t + \tilde{A}_{12} Z_{2t-1}.$$

It is easy to check that the controlled system will have the following form

$$(5.25)\qquad \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} \{I + \tilde{R} K_M\}^{-1} \tilde{A}_{11} & \tilde{A}_{21} \\ \tilde{A}_{12} & \tilde{A}_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix}$$

where
$$\underset{l \times l}{\tilde{R}} = \begin{array}{cc|c} \begin{pmatrix} R^{-1} & 0 \\ 0 & 0 \end{pmatrix} & & \begin{array}{c} m \\ l-m \end{array} \\ \begin{array}{cc} m & l-m \end{array} & & \end{array}$$

Clearly the controlled system (5.25) is asymptotically stable if the off-diagonal blocks $\tilde{A}_{12}$ and $\tilde{A}_{21}$ are satisfied with the appropriate sufficiently smallness condition. As an extreme case, all the characteristic roots are less than

1 if the off-diagonal blocks are identically zero. In general, it will be desirable if the controlled system has the property of a sequentially decreasing quadratic loss function, by the Lyapunov function theorem in Chapter II. Now let us consider what conditions the quadratic loss function constructs the sequentially decreasing sequence. Recalling $Z_t=S^{-1}y_t$ in Section 1, the quadratic loss function (5. 1) can be rewritten as

$$(5.\,1)' \qquad W_\infty=\sum_{t=1}^{\infty} Z'_{t-1}\,S'\,QS\,Z_{t-1}+x_t'\,R\,x_t$$

$$=\sum_{t=1}^{\infty} Z'_{t-1}\,S'\,QS\,Z_{t-1}+Z'_{1t-1}\,\tilde G'\,R\,\tilde G\,Z_{1t-1}$$

because $x_t=-\tilde G\,z_{1t-1}$ from equation (5. 25), then

$$(5.\,1)'' \qquad W_\infty=\sum_{t=1}^{\infty} Z'_{t-1}\,\tilde Q\,z_{t-1}$$

where
$$\tilde Q=S'\,Q\,S+\begin{array}{cc} \left(\begin{array}{cc} \tilde G\,R\,\tilde G & 0 \\ 0 & 0 \end{array}\right) & \begin{array}{c} l \\ n-l \end{array} \\ \begin{array}{cc} l & \quad n-l \end{array} & \end{array}$$

Separating the equation (5. 25) into two parts,

$$(5.\,25)' \qquad Z_t=\tilde H\,Z_{t-1}+\hat H\,Z_{t-1}$$

where
$$\tilde H=\begin{pmatrix} (I+\tilde R\,K_M)^{-1}\tilde A_{11} & 0 \\ 0 & \tilde A_{22} \end{pmatrix} \text{ and } \tilde H=\begin{pmatrix} 0 & \tilde A_{12} \\ \tilde A_{21} & 0 \end{pmatrix}$$

Furthermore, let us make the similar assumption for (5. 23) for simplicity,

$$(5.\,23)' \qquad |\lambda_{\max}(\tilde A_{22})|<1$$

Then there exists the positive definite matrix $\tilde K$ such that

$$(5.\,26) \qquad \underset{n\times n}{\tilde H\,\tilde K\,\tilde H}-\tilde K=\underset{n\times n}{-\tilde Q}$$

Furthermore,

$$(5.\,27) \qquad \tilde K=\begin{array}{cc} \left(\begin{array}{c|c} K_M & 0 \\ \cdots & \cdots \\ 0 & 0 \end{array}\right) & \begin{array}{c} l \\ \\ n-l \end{array} \\ \begin{array}{cc} l & n-l \end{array} & \end{array}$$

which is clear from equation (5. 25) and Chapter II. Then the loss function (5. 1)″ at time $t$, $W_{t,\infty}$, can be written by

$$(5.\,28) \qquad W_{t,\infty}=Z'_{t-1}\,\tilde K\,Z_{t-1}.$$

when $\tilde{A}_{12}$ and $\tilde{A}_{21}$ are identically equal to zero. Now we want to show that even if $\tilde{A}_{12}$ and $\tilde{A}_{21}$ are non-zero, $\triangle W_{t+1,\infty}=W_{t+1,\infty}-W_{t,\infty}<0$, which is the basic property of stability in the sense of Lyapunov, if $\tilde{A}_{12}$ and $\tilde{A}_{21}$ satisfy the certain sufficient smallness condition.

From equation (5. 28)

$$\triangle W_{t+1,\infty}=Z_t'\,\tilde{K}\,Z_t-Z'_{t-1}\,\tilde{K}\,Z_{t-1} \tag{5. 29}$$

From equation (5. 26) and equation (5. 25)′, equation (5. 29) can be rewritten as

$$\begin{aligned}\triangle W_{t+1,\infty}=&-Z'_{t-1}\,\tilde{Q}\,Z_{t-1}+Z'_{t-1}\,\hat{H}'\,\tilde{K}\,\tilde{H}\,Z_{t-1}\\&+Z_{t-1}\,\hat{H}'\,\tilde{K}\,\hat{H}\,Z_{t-1}\end{aligned} \tag{5. 29)′}$$

Using the Schwartz inequality and dividing by $W_{t,\infty}$,

$$\frac{\triangle_{t+1,\infty}}{W_{t,\infty}}=\frac{-Z'_{t-1}\,\tilde{Q}\,Z_{t-1}+\|Z_{t-1}\|\;\|\hat{H}\|\;\|\tilde{K}\,\tilde{H}\,Z_{t-1}\|}{Z'_{t-1}\,\tilde{K}\,Z_{t-1}}$$
$$\frac{+\|H\|^{2}\,\|\tilde{K}\,\|Z_{t-1}\|^{2}}{} \tag{5. 30}$$

where $\|\cdot\|$ denotes Euclidean norm.

We have to note that $\triangle W_{t+1,\infty}<0$ if and only if $\dfrac{\triangle W_{t+1,\infty}}{W_{t,\infty}}<0$ because $W_{t,\infty}>0$.

Now we want to show that the right-hand side of inequality (5. 30) is negative for all $Z_{t-1}$ under the certain condition on $\|H\|$.

(Theorem 4) $\triangle W_{t+1,\infty}<0$ if

$$\|H\|\leqq\frac{-C_2+\sqrt{C_2{}^2+\eta\,C_3{}^2}}{C_1\,C_3{}^2} \tag{5. 31}$$

where
$$C_1=\{\lambda_{\min}(\tilde{K})\}^{-1/2}$$
$$C_2=\{\lambda_{\max}(\tilde{H}'\tilde{K}^2\tilde{H}\,\tilde{K}^{-1})\}^{1/2}$$
$$C_3=\ \lambda_{\max}(\tilde{K})^{1/2}$$
$$z=\ \lambda_{\min}(\tilde{Q}\,\tilde{K}^{-1})$$

(Proof) From equation (5. 30),

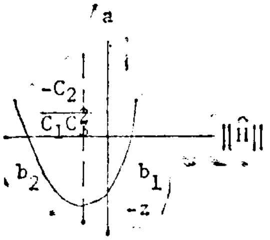
$$(5.30)' \qquad \frac{\triangle W_{t+1,\infty}}{W_{t,\infty}} < -\frac{Z'_{t-1}\tilde{Q}Z_{t-1}}{Z'_{t-1}\tilde{K}Z_{t-1}} + 2\,\|\hat{H}\|\left(\frac{Z'_{t-1}Z_{t-1}}{Z'_{t-1}\tilde{K}Z_{t-1}}\right)^{1/2} \cdot \left(\frac{Z'_{t-1}\tilde{H}'\tilde{K}^2\tilde{H}Z_{t-1}}{Z'_{t-1}\tilde{K}Z_{t-1}}\right)^{1/2} + \|\hat{H}\|^2\,\frac{Z'_{t-1}Z_{t-1}}{Z'_{t-1}\tilde{K}Z_{t-1}}\,\|\tilde{K}\|$$

By the definition of Euclidean norm, which is defined as the square root of the inner product. Maximizing each term of the right-hand side in inequality

$$(5.30)'' \qquad \frac{\triangle W_{t+1,\infty}}{W_{t,\infty}} < -\eta + 2\,\|\hat{H}\|\,C_1C_2 + \|\hat{H}\|^2\,C_1^2C_2^3$$

In order that $\frac{\triangle W_{t+1,\infty}}{W_{t,\infty}}$ be negative, $a = -\eta + 2\,\|\hat{H}\|\,C_1C_2 + \|\hat{H}\|^2\,C_1^2C_3^2$ should be negative with respect to some $\|\hat{H}\|$. We have to note that the quadratic equation, $\|\hat{H}\|^2\,C_1^2C_3^2 + 2\,\|\hat{H}\|\,C_1C_2 - \eta = 0$, has two distinct roots, because the determinant $C_1^2C_2^2 + \eta\,C_1^2C_3^2$ is positive, Furthermore, two distinct roots have opposite signs and the negative root is not interested because $\|\hat{H}\|$ is positive. Therefore it is easy to see from the following graph that a would be negative if $\|\hat{H}\|$ is less than

$$\frac{-C_2 + \sqrt{C_2^2 + \eta\,C_3^2}}{C_1C_3^2}$$



$$b_1 = \frac{-C_2 + \sqrt{C_2^2 + \eta\,C_3^2}}{C_1C_3^2}$$

$$b_2 = \frac{-C_2 - \sqrt{C_2^2 + \eta\,C_3^2}}{C_1C_3^2}$$

q.e.d.

In fact, inequality (5.31) is so complicated that it is very hard to interpret and, in reality, it is not easy to check whether the inequality holds or not. All we can do in order to check condition (5.31) is the approximate calculation of $\triangle W_{t+1,\infty}$ by the dynamic simulation of the controlled system (5.25) with a fairly long-time period. Furthermore, we have to note that the inequality (5.31) is an a posteriori condition, which cannot be detected as an a priori condition, because $\tilde{K}$ is unknown a priori. Similar phenomena appear in the adaptive control rule of Chapter II. This is the very reason why we have mentioned the importance of heuristic criteria, as far as the aggregation is concerned, at the beginning of this chapter.

So far, we have discussed the analysis of the canonical representation. But Theorem 4 can be extended to a much more general form of the model like the (4. 1) with any kind of scheme of aggregation. Suppose that aggregation matrix be constructed by

$$(5.32) \qquad C = l\{\, I \mid \underset{n-l}{0} \,\}$$

Defining the aggregative variables,

$$(5.33) \qquad z_t = Cy_t = y_{1t}$$ the matrix equation.

From the least square solution of $CA = FC$,

$$(5.34) \qquad F = CAC'(CC')^{-1} = Z_{11}$$

The aggregative model will be

$$(5.35) \qquad y_{1t} = A_{11} y_{1t-1} + B_1 x_t + V_{1t}$$

where $V_{1t}$ denotes the aggregation error which is nothing but $A_{12} y_{2t-1}$.

The quadratic loss function will be

$$(5.36) \qquad \tilde{W} = \sum_{t=1}^{\infty} (y_1{}'_{,t-1} Q_{1t} y_{1t-1} + x_t{}' R x_t)$$

where

$$Q = \begin{array}{cc} \left(\begin{array}{c|c} Q_{11} & Q_{12} \\ \cdots\cdots & \cdots\cdots \\ Q_{21} & Q_{22} \end{array}\right) & \begin{array}{l} l \\ \\ n-l \end{array} \\ \begin{array}{cc} \quad l & \quad n-l \end{array} & \end{array}$$

Suppose the control equation be derived as

$$(5.37) \qquad x_t = -(R + B_1{}' K_M B_1)^{-1} B_1{}' K_M A_{11} y_{1t-1} = -G y_{1t-1}$$

Then the controlled system will be

$$(5.38) \qquad \begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} A_{11} - B_1 G & A_{12} \\ A_{21} - B_2 G & A_{22} \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix}$$

Now we can analyze equation (5. 38) in analogy the Theorem 4, when $A_{22}$ is assumed to have characteristic roots within the unit circle. Defining

$$(5.39) \qquad \tilde{H} = \begin{pmatrix} A_{11} - B_1 G & 0 \\ 0 & A_{22} \end{pmatrix}$$ and

$$(5.40) \qquad \tilde{H} = \begin{pmatrix} 0 & A_{12} \\ A_{21} - BG & 0 \end{pmatrix}$$

We could derive the similar condition as inequality (5. 31). The difficulty with equation (5. 40) is that the left bottom block of the error matrix, $\hat{H}$, includes the matrix $G$ which is unknown a priori. Thus we have to be satisfied simply with the sufficiently smallness of the matrices $A_{12}$, $A_{21}$ and $B_2$ by the same reasoning as the previous analysis. Again we could rely upon dynamic simulation of equation (5. 38) in order to check whether the sufficiently smallness condition is satisfied or not. In other words, if the sequences of the loss function $W_{t:N}$ computed from the dynamic simulation for the moving longtime horizon $N$ is decreasing, we could infer that those conditions on the sufficient smallness, described in Thecrem 4, are satisfied and the controlled system is stable.

## 6. Conclusion and Future Research

This chapter discusses the reduction of the dimension of the model by aggreagation and the control of the aggregated model. We have distinguished between long-run policy. The purpose of the long-run policy should be in the stabilization of the economic system and the purpose of the short-run policy should be in achieving target-levels. Long-run policy is determined by the coefficient matrix of the lagged endogenous variables in the linear feedback equation and short-run policy is determined by the intercept term. By and large, the long-run policy is to shift the characteristic roots of the econometric model within the unit circle. Short-run policy should be determined by the optimization techique which, though the model is fairly large, are computationally feasible. Furthermore, we have observed in the equation (5. 1) that the long-run policy problem can be transformed into the short-run policy problem with the appropriate choice of the terminal conditions. It has been shown in Chapter II, that the coefficient matrix of the lagged endogenous variables in the linear feedback equation, which determines long-run policy, does not depend upon the target-level. From those findings, we have suggested a two stage computation in Section 5 for the design of the control strategy on the unstable econometric model, even if it is not optimal, to fulfill approximately the long-run and short-run policies as well. The first stage is to determine the approximate steady-state solution of the Ricatti difference equation to get the matrix $K$ such that the coefficient matrix of the controlled system, $A-BG$, has characteristic roots within the unit circle. The second stage is to determine short-run policy through the optimization technique, using such the $K$ obtained at the first stage as a terminal condition weight matrix of the target variables with a finite time horizon, The determination of the short-run policy through the optimization technique in the second stage was not discussed in this chapter, because it has been studied by many economists so far. The main purpose of aggregation is to determine the steady-state matrix solution $K$ of the Ricatti difference equation approximately in the first stage for the long-run policy, but not for short-run policy. Ando-Simon have shown that the steady-state solution of the linear difference equation can be approximated by the characteristic vectors corresponding to the largest characteristic roots. But we have to note that the matrix Ricatti difference equation is not linear with respect to the matrix $K$. Thus the change rate of $K$ is not constant, but time-varying. It is easy to see the Ricatti difference equation (1. 7) of the Chapter III can be rewritten, in terms of the change of $K$, as

$$\begin{aligned}(6.1)\qquad \delta K_t = &\{(I+BR^{-1}B'K_{t+1})^{-1}A\}'\,\delta K_{t+1}\\ &\{(I+BR^{-1}B'K_t)^{-1}\}A\end{aligned}$$

where $\delta$ denotes the backward difference operator, $\delta K_t = K_{t-1} - K_t$ because the dynamic programming algorithm is computed by backward recursiveness. The coefficient matrix of the equation (6. 1) includes the $K_t$ which is unknown a priori. Therefore, the aggregation of Ando-Simon by the characteristic vectors corresponding to the largest characteristic roots and the control parameter $K$, cannot be separated, but interact with each other. Considering the main purpose of long-run control policy should be in stabilizing of the economic system, or shifting the characteristic roots of A into the unit circle by the control policy, we have shown in Section 5 that aggregation by the characteristic vectors corresponding to the largest characteristic roots is sufficient for the stabilization qolicy. This control may not be optimal, but will stabilize the econometric model, which implies the deviation between the true optimal cost and the approximal cost is finite, In reality, the characteristic vectors are difficult to compute in general and the canonical representation of Section 1 is not easy to derive. Under these circumstances, we have discussed the approximate derivation of the model which has the largest characteristic roots of the original model, using such the a priori knowledge as in the nearly complete decomposability or weak coupling. We have also discussed the effect of control in the approximately derived aggregative model on the stability of the controlled economic system and in Theorem 4, we have derived the sufficient condition of the approximation error to guarantee stability. We have claimed that this condition might not be useful from a practical point of view for empirical research. So it has been suggested to check the sequence of the numerically computed loss functional which can be computed by simulating of the original model and the control equation together with the moving time horizon. If the sequence is decreasing, the controlled economic system would be asmptotically stable by the Lyapunov function theorem. However, Theorem 4 has broad applicability. In other words, the error martxi of the equation (5. 31), $H$, may include the error from misspecification, from the estimation, or from structural shift. Furthermore, if the nonlinear model is linearized, it could include the error from the linear approximation. So far we have discussed the application of the aggregation concept to the control strategy to determine the steady-state solution of the Ricatti difference equation, $K$, approximately so that the resulting controlled system could have the asymptotic stability property in the Lyapunov sense. Thus the matrix solution, $K$, is utilized as a terminal condition of the quadratic losss function for the short-run policy. We could consider the approximate computation of the matrix $K$ in another way for future research. We have proposed a two-stage computation scheme both for long-run policy and for short-run policy at the beginning of this section. At the first stage, the matrix $K$ is computed approximately to derive the coefficient matrix $G_M$, of the lagged aggregated endogenous variables of the equation (5. 9). The procedure of the other direction can be considered by the similar reasoning of approximation via aggregation. (1) the computation of $G$ such that

$$|\lambda(A-BG)|<1 \tag{6. 2}$$

and then (2) solving the matrix equation of Theorem 1 in Chapter II,

$$(A-BG)'K(A-BG)-K=-(Q+G'RG), \tag{6. 3}$$

to get the solution, $K$, which will be used as a terminal condition for the short-run policy in the second stage. If we could find the matrix $G$ such that the inequality (6. 2) would be satisfied, then the matrix equation (6. 3) could be solved by the simple iterations.

Rewriting the equation (6. 3)

$$K_{h+1}=Q+G'RG+(A-BG)'K_h(A-BG) \tag{6. 4}$$

where $h$ denotes the hth iteration.

The iterative matrix equation (6. 4) converges to the finite solution $K$ because $|\lambda(A-BG)|<1$. The problem is how the matrix $G$ could be found such that $|\lambda(A-BG)|<1$. From this author's knowledge, there is no general method to find the matrix $G$ such that $|\lambda(A-BG)|<1$.

Now we are going to present as an application of the theorem on dominant diagonal matrices from economic theory. McKenzie(13) has proved the following theorem on the stability of the general equlibrium model.

(Definition) An $n\times n$ matrix $A=(a_{ij})$ is said to have a dominant diagonal if there exist $w_i>0$ such that $w_i|a_{ii}|>\sum_{i\neq j} w_j|a_{ji}|$ for each $i$. If $a_{ii}$ are positive for all $i$, $A$ is said to have a positive dominant diagonal.

(Theorem 5) For $H$, a non-negative square matrix, a necessary and sufficient condition that all the characteristic roots of $H$ lie within the unit circle is that $(I-H)$ has a positive dominant diagonal.

(Proof) See L. McKenzie(13).

As a simple application of Theorem 5 by McKenzie, we could have a weaker result when $H$ is not nonnegative.

(Corollary 9) For any square matrix, $H=(h_{ij})$, define an equimodular set of matrices as $H^*=(|h_{ij}|)$(18). A sufficient condition that all characteristic roots of $H$ lie within the unit circle is that $(I-H^*)$ has a positive dominant diagonal.

(Proof) See Baer (7) or see Marcus and Ming (14).

For the application of the Corollary 6, $A-BG$ can be rewritten as

$$A-BG=(a_1,a_2,\dots,a_n)-\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}(G_1,G_2,\dots,G_n) \tag{6.5}$$

where $A=(a_1,a_2,\dots,a_n)$ for $n\times 1$ column vector $a_1$

$$B=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad \text{for } 1\times m \text{ row vector } b_1$$

and $G=(G_1,G_2,\dots,G_n)$ for $m\times 1$ column vector $G_1$

Let an arbitrary weight matrix $w$ be

$$w=\begin{pmatrix} w_1\cdots 0 \cdots 0 \\ \vdots \quad\quad\quad \vdots \\ 0 \quad w_2 \quad \vdots \\ \vdots \quad\quad\quad \vdots \\ 0 \cdots\cdots\cdots w_n \end{pmatrix} \qquad \text{For } w_1>0 \tag{6.6}$$

Define the equimodular set of matrices of $A$ as

$$A^*=\{|a_{ij}|\}=\{a_1^*,a_2^*,\dots,a_n^*\} \tag{6.7}$$

Without loss of generality, suppose that the last $(n-l)$ columns of $A^*$ are positive dominant diagonal or

$$w_i(I-|a_{ii}|)>\sum_{i\neq j} w_j|-a_{ij}| \tag{6.8}$$

$$i=l+1,\dots,n \text{ and } j=1,2,\dots,n$$

Then it is easy to see that matrix $G$ could be constructed by

$$G=\{G_1,G_2,\dots G_l,\ 0\dots 0\} \tag{6.9}$$

Now let us consider the ith column of $A-BG$,

$$(6.10) \qquad (A-BG)_i = a_i - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} G_i \qquad \text{for } i=1,\ldots,l$$

From the definition of the dominant diagonal we can compute $G$ such that $|\lambda(A-BG)|<1$,

$$(6.11) \qquad w_i(1-|a_{ii}-b_iG_i|) > \sum_{i \neq 1} w_j|-a_{ji}+b_jG_i|$$

$$= \sum_{j \neq 1} w_j|a_{ji}-b_jG_i|$$

$$i=1,\ldots,l \text{ and } j=1,\ldots,n$$

Rewriting inequality (6. 1),

$$(6.12) \qquad w_i > \sum_{j=1}^{n} w_j|a_{ji}-b_jG_i|$$

Clearly inequality (6. 12) is a sufficient condition the $|\lambda(A-BG)|<1$, In fact, it is not easy to find the $m \times 1$ column vector $G_i$ to satisfy the inequality because the right-hand side of inequality (6. 12) is a sum of absolute values. A sufficient condition that the inequality be satisfied has to be considered for the practical application. From the following inequality,[3]

$$(6.13) \qquad n\sum_{j=1}^{n} w_j\{(a_{ji}-b_jG_i)\}^2 \geq (\sum_{j=1}^{n} w_j|a_{ji}-b_jG_i|)^2,$$

a sufficient condition for the inequaliy (6. 12) may be rewritten as

$$(6.14) \qquad w_i^2 > n\sum_{j=1}^{n} \{w_j(a_{ji}-b_jG_i)\}^2$$

cr

$$\frac{W^2{}_i}{n} > (a_i-BG_i)'w^2(a_i-BG_1)$$

The minimum of the right-hand side of inequality (6. 14) can be interpreted as Aitken's generalized least square where $G_i$ is a column vector of the parameters to be determined. From the formula of the generalized lease square,

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[3]
$$n\sum_{j=1}^{n} X_j^2 - (\sum_{j=1}^{n} X_j)^2 = \sum_{j>i}(X_i-X_j)^3 \geq 0$$

or

$$n\sum_{j=1}^{n} X_j^2 \geq (\sum_{j=1}^{n} X_j)^2$$

$$(6.\ 15)\qquad G_i = B'w^2B)^{-1}B'w^2a_i$$

Substuting equation (6. 15) into inequality (6. 14) and using the idempotency of $(I-B(B'w'B)^{-1}B'w^2)$, we can rewrite inequality (6. 14) as

$$(6.\ 16)\qquad \frac{w_i^2}{n} > a_i'\Omega a_i ; i=1, 2, \ldots, l$$

where $\Omega = w^2(I-B(B'w^2B)^{-1}B'w^2)$.

Clearly inequality (6. 16) is a sufficient condition that $|\lambda(A-BG)|<1$. Moreover, this is easy to test if the column of $B$ or the number of instruments are fairly small enough. In case of the distributed lag model, the foregoing ad hec method for stabilization is not easy to apply in general. Consider the state variable form of the distributed lag model.

$$(6.\ 17)\qquad \underset{(nk)\times 1}{\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-k+1} \end{bmatrix}} = \underset{(nk)\times(nk)}{\begin{bmatrix} A_1 & A_2 & A_i \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I0 \end{bmatrix}} \underset{(nk\times 1)}{\begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-k} \end{bmatrix}} + \underset{(nk)\times m}{\begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}} \underset{m\times 1}{x_t}$$

When the control equation is in the form of

$$(6.\ 18)\qquad x_t = -(G_1 y_{t-1} + G_2 y_{t-2} + \ldots G_k y_{t-k}),$$

the controlled system will be

$$(6.\ 19)\qquad \begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-k+1} \end{bmatrix} = \begin{bmatrix} A-BG_1 & A_2-BG_2 \cdots A_k-BG_k \\ I & 0 \quad \cdots \quad 0 \\ 0 & I \quad \cdots \quad 0 \\ 0 & 0 \quad \cdots \quad I0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-k} \end{bmatrix}$$

where $G_i$'s are $m\times n$ matrices.

It is very difficult to find out the $nk\times nk$ weight matrix, $w$. It should make as many columns of the dominant diagonal of the coefficient matrix of the lagged endogenous variables in equatiaion (6. 17) as possible in theory, we could choose some weight matrix as

$$(6.\ 20)\qquad \underset{nk\times nk}{w} = \begin{bmatrix} w_1 & 0 \cdots 0 \\ 0 & w^2 \cdots 0 \\ \vdots & \vdots \quad\ \ \vdots \\ 0 & 0 \cdots w^k \end{bmatrix} ; w_i > w_i^{l+1} > 0$$

where $w^j$ is a $n\times n$ diagonal matrix and $w_i^j$ is the ith diagonal element of $w^j$.

By an analogous analysis to the single lag case, we can derive a sufficient condition for the stabilizability, as in inequality (6. 12), which is

(6.21) $$w_i^l > \sum_{i=1}^{n} w_i^1 |a^l_{ij} - b_i G_j^l| + w_i^{l+1}$$

or $$w_i^l - w_i^{l+1} > \sum_{i=1}^{n} w_i^1 |a^l_{ij} - b_i G_i^l|$$

where $a^l_{ij}$ is the $i$, jth element of $A_l$,

$b_i$ is the ith row vector, and

$G_j^l$ is the jth colum vector of $G_l$.

In the analogy to the foregoing analysis, we could derive another sufficient condition as

(6.22) $$\frac{w_i^l - w_i^{l+1,2}}{n} = a_i^{l\prime} \Omega a_i^l$$

where $\Omega = (w^1)^2 (I - B(B(w^1)^2 B)^{-1} B'(w^1)^2)$

and $a_i^l$is the ith column vector of $A_l$.

and

(6.23) $$G_i^l = (B'(w^1)^2 B)^{-1} B'(w^1)^2 a_i^l$$

where $G_i^l$ is the ith column vector of $G_l$.

Again the generalized least square interpretation of (6.22) and (6.23) implies

(6.24) $$\underset{G_i l}{\mathrm{Min}} (a_i^l - BG_i^l)'(w^1)^2 (a_i^l - BG_i^l).$$

From expression (6.24), it is easy to see that, given an arbitrary weight matrix $w^1$, the same analysis as in the single lag case can be done with respect to each lagged coefficient matrix $A_l$. Another useful inequality to satisfy this inequality is

(6.25) $$w_i^l - w_i^k > \sqrt{n} \sum_{l=1}^{k} (a_i^{l\prime}\ \Omega\ a_i^l)^{1/2}.$$

Because we can consider $w_i^k$ to be arbitrarily small, the inequality (6.25) can be rewritten as

(6.26) $$\frac{(w_i^1)^2}{n} > \{\sum_{l=1}^{k} (a_i^{l\prime}\ \Omega\ a_i^l)^{1/2}\}2$$

Comparing (6.16) and (6.26), the right-hand sides of both ineqalities have the similar expression. As mentioned before, this is not a general method, but an ad hoc method oriented to practical applications. Thus some research remains to be done for the equation (6.2). If not, some method of choosing the weight matrix $w$ must be devloped. Future research in this area is potentially useful. Another interesting point of the above ad hoc method is that the weight matrix $w$ has the similar characteristics with the matrix $K$ of the Ricatti difference equation, except that $w$ has been restricted to the positive definite diagonal matrix while $K$ being the general positive definite symmetric matrix.

What is the most important for the futur research is the control of the structural form prior to the transformation to the reduced form because the derivation of the reduced form of the large scale model is very difficult. Most of the methods discussed in this chapter assumes the model is in the reduced form, though we did discuss the structural model in Section 4 (see the equation (4.23) by means of equation (4.31). If we consider the approximate derivation of the reduced form 4 might be very useful. Thus Chapter 5 has focused on the application of Theorem 4 in its generalized form when the reduced form is unknown. The unknown reduced form may happen, due to nonlinearity in the structural form or due to computational difficulties in the linear large-scale model.

As we mentioned in the introduction of this chapter, all the analysis of this chapter may be directly applied to stochastic models with additive error terms of white noise. The aggregation method in the stochastic coefficient case remains to be solved. Example 3 of Section 3, which is a method suggested by $W$. Fisher (8) may be useful, because the quadratic criteria of the loss function of the aggregation can be considered as the expected value, or $EL=tr\ E(\tilde{F}-A)'(\tilde{F}-A)$ From equation (3.19).

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